SUBMANIFOLDS WITH CONSTANT JORDAN ANGLES

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Abstract. To study the Lawson-Osserman's counterexample [26] to the Bernstein problem for minimal submanifolds of higher codimension, a new geometric concept, submanifolds in Euclidean space with constant Jordan angles(CJA), is introduced. By exploring the second fundamental form of submanifolds with CJA, we can characterize the Lawson-Osserman's cone from the viewpoint of Jordan angles.

Contents

1. Introduction

In previous works, we have systematically studied the Bernstein problem for complete minimal submanifolds of higher codimension in Euclidean space (see [19, 21, 22, 24, 34]). In particular, we could prove that a complete minimal submanifold in Euclidean space is affine linear if it does not deviate too much from a linear

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subspace in the sense that a certain function v defined in terms of Jordan angles is bounded by 3. It is natural to ask whether that number is optimal. Now, there is the Lawson-Osserman's counterexample [26] to the higher codimension Bernstein problem for which v is identically 9. The aim of the present paper then is to understand this example in geometric terms, in particular in terms of Jordan angles. Here, the Jordan angles between two linear subspaces P and Q are the critical values of the angle θ between the nonzero vectors u in P and their orthogonal projection u^* in Q. When these Jordan angles are constant for all the normal spaces of some submanifold M of Euclidean space and a fixed linear reference subspace, we say that M has constant Jordan angles. This is the fundamental concept of our paper, and we abbreviate it as CJA. For a precise statement, refer to Definition 1.1 below. Now it turns out the Lawson-Osserman's counterexample has CJA relative to the imaginary quaternions when viewed as a subspace of the imaginary octonians. Harvey-Lawson [18] showed that the Lawson-Osserman's cone is a four dimensional coassociative submanifold in $\mathbb{R}^7$ which can be identified with the imaginary octonians. Therefore, we study such coassociative submanifolds with CJA and find that a coassociative graph with CJA relative to the imaginary quaternions and at most two different normal Jordan angles either is affine linear or a translate of a portion of the Lawson-Osserman's cone.

For more precise statements, we now develop some notation and technical concepts.

1.1. **Jordan angles and angle spaces.**

Let P and Q_0 be m-dimensional subspaces (i.e. m-planes) in $\mathbb{R}^{n+m}$. The *Jordan angles* between P and Q_0 are the critical values of the angle θ between a nonzero vector u in P and its orthogonal projection u^* in Q_0 as u runs through P. This concept was firstly introduced by Jordan [20] in 1875, and they are also called *principal angles* in some references, e.g. [13]. If θ is a nonzero Jordan angle between P and Q_0 determined by a unit vector u in P and its projection u^* in Q_0, then u is called an *angle direction* of P relative to Q_0, and the 2-plane spanned by u and u^* is called an *angle 2-plane* between P and Q_0 (see [31]).

Denote by $\mathcal{P}_0$ the orthogonal projection of $\mathbb{R}^{n+m}$ onto Q_0 and by $\mathcal{P}$ the orthogonal projection of $\mathbb{R}^{n+m}$ onto P. Then for any $u \in P$ and $\varepsilon \in Q_0$,

$$\begin{aligned}\langle \mathcal{P}_0 u, \varepsilon\rangle &= \langle \mathcal{P}_0 u + (u - \mathcal{P}_0 u), \varepsilon\rangle = \langle u, \varepsilon\rangle \\ &= \langle u, \mathcal{P}\varepsilon + (\varepsilon - \mathcal{P}\varepsilon)\rangle = \langle u, \mathcal{P}\varepsilon\rangle\end{aligned} \tag{1.1}$$

and moreover

$$\langle (\mathcal{P}\circ\mathcal{P}_0)u, v\rangle = \langle \mathcal{P}_0 u, \mathcal{P}_0 v\rangle = \langle u, (\mathcal{P}\circ\mathcal{P}_0)v\rangle \tag{1.2}$$

holds for every $u, v \in P$, which implies $\mathcal{P}\circ\mathcal{P}_0$ is a nonnegative definite self-adjoint transformation on P.

For any nonzero vector $u \in P$,

$$\cos^2 \angle(u, u^*) = \frac{\langle u^*, u^*\rangle}{\langle u, u\rangle} = \frac{\langle \mathcal{P}_0 u, \mathcal{P}_0 u\rangle}{\langle u, u\rangle} = \frac{\langle (\mathcal{P}\circ\mathcal{P}_0)u, u\rangle}{\langle u, u\rangle}. \tag{1.3}$$

Hence θ is a Jordan angle between P and Q_0 if and only if $\mu := \cos^2\theta$ is an eigenvalue of $\mathcal{P}\circ\mathcal{P}_0$, and u is an angle direction with respect to θ if and only if u is an eigenvector associated to the eigenvalue μ, i.e.

$$(\mathcal{P}\circ\mathcal{P}_0)u = \mu u = \cos^2\theta\ u. \tag{1.4}$$

Therefore, all the angle directions with respect to θ constitute a linear subspace of P, which is called an *angle space* of P relative to Q_0 and we denote it by P_θ. In particular,

$$P_0 = P\cap Q_0, \qquad P_{\pi/2} = P\cap Q_0^\perp. \tag{1.5}$$

The dimension of P_θ is called the *multiplicity* of θ, which is denoted by m_θ. If we denote by $\mathrm{Arg}(P,Q_0)$ the set consisting of all the Jordan angles between P and Q_0, then

$$P = \bigoplus_{\theta\in\mathrm{Arg}(P,Q_0)} P_\theta \tag{1.6}$$

and the angle spaces are mutually orthogonal to each other. Hence

$$m = \sum_{\theta\in\mathrm{Arg}(P,Q_0)} m_\theta. \tag{1.7}$$

The Jordan angles between two m-planes completely determine their relative positions. More precisely, one can conclude that:

Proposition 1.1. [31] *Let P_1, Q_1 and P_2, Q_2 be any two pairs of m-planes in $\mathbb{R}^{n+m}$. If $Arg(P_1,Q_1) = Arg(P_2,Q_2)$ and the multiplicities of the corresponding Jordan angles are equivalent, then there exists a rigid motion of $\mathbb{R}^{n+m}$, carrying P_1, Q_1 onto P_2, Q_2, respectively. And vice versa.*

Similarly, let $\mathrm{Arg}(Q_0,P)$ denote the set consisting of all the Jordan angles between Q_0 and P, then $\theta\in\mathrm{Arg}(Q_0,P)$ if and only if $\mu := \cos^2\theta$ is an eigenvalue of $\mathcal{P}_0\circ\mathcal{P}$. Denote by $(Q_0)_\theta$ the angle space of Q_0 relative to P associated to θ, then $\varepsilon\in(Q_0)_\theta$ if and only if $(\mathcal{P}_0\circ\mathcal{P})\varepsilon = \cos^2\theta\ \varepsilon$, and

$$Q_0 = \bigoplus_{\theta\in\mathrm{Arg}(Q_0,P)} (Q_0)_\theta. \tag{1.8}$$

Let $P^\perp$ and $Q_0^\perp$ be the orthogonal complements of P and Q_0, and denote by $\mathcal{P}^\perp$ and $\mathcal{P}_0^\perp$ the orthogonal projections of $\mathbb{R}^{n+m}$ onto $P^\perp$ and $Q_0^\perp$, respectively. As above, the set consisting of all the Jordan angles between $P^\perp$ and $Q_0^\perp$ is denoted by $\mathrm{Arg}(P^\perp,Q_0^\perp)$, $P_\theta^\perp$ denotes the angle space associated to $\theta\in\mathrm{Arg}(P^\perp,Q_0^\perp)$, and $m_\theta^\perp := \dim P_\theta^\perp$ denotes the multiplicity of θ.

The following lemma reveals the close relationship between $\mathrm{Arg}(P,Q_0)$, $\mathrm{Arg}(Q_0,P)$ and $\mathrm{Arg}(P^\perp,Q_0^\perp)$.

Lemma 1.1. *([24]) Let P, Q_0 be m-planes in $\mathbb{R}^{n+m}$, then $Arg(P,Q_0) = Arg(Q_0,P)$ and the multiplicities of each corresponding Jordan angles are equivalent. If we denote*

$$R_\theta := P_\theta + (Q_0)_\theta \tag{1.9}$$

for each $\theta \in Arg(P, Q_0)$, then $R_\theta \perp R_\sigma$ whenever $\theta \neq \sigma$, and

$$P + Q_0 = \bigoplus_{\theta \in Arg(P,Q_0)} R_\theta. \tag{1.10}$$

For any $\theta \in (0, \pi/2]$, $\theta \in Arg(P^\perp, Q_0^\perp)$ if and only if $\theta \in Arg(P, Q_0)$, and $m_\theta^\perp = m_\theta$, $R_\theta = P_\theta \oplus P_\theta^\perp$. Moreover, for every $\theta \in Arg(P, Q_0) \cap (0, \pi/2)$, there exists an isometric automorphism $\Phi_\theta : R_\theta \to R_\theta$, such that

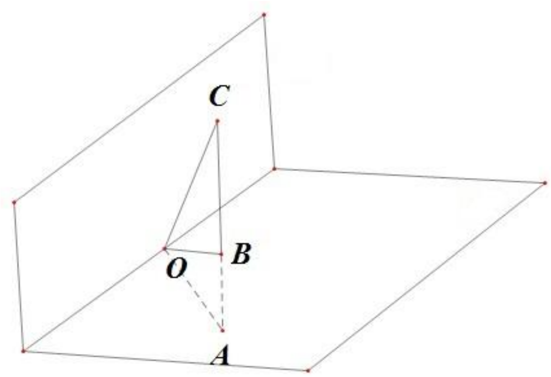
(i) $\Phi_\theta(P_\theta) = P_\theta^\perp$, $\Phi_\theta(P_\theta^\perp) = P_\theta$;

(ii) $\Phi_\theta^2 = -\mathbf{Id}$;

(iii) For any nonzero vector $u \in P_\theta$ ($v \in P_\theta^\perp$), $\Phi_\theta(u)$ ($\Phi_\theta(v)$) lies in the angle 2-plane generated by u (v); more precisely,

$$\begin{aligned} \sec\theta\ \mathcal{P}_0 u &= \cos\theta\ u - \sin\theta\ \Phi_\theta(u), \\ \sec\theta\ \mathcal{P}_0^\perp v &= \cos\theta\ v - \sin\theta\ \Phi_\theta(v). \end{aligned} \tag{1.11}$$

Remark. Let P and Q_0 be a pair of intersecting planes in $\mathbb{R}^3$, then $\mathrm{Arg}(P, Q_0) = \{\theta, 0\}$, where θ is the dihedral angle between P and Q_0.



Denote by l their line of intersection and O the origin of $\mathbb{R}^3$. Choose $A \in \mathbb{R}^3$, such that $v := \overrightarrow{OA}$ is a unit vector orthogonal to P. Through A, draw a perpendicular line to Q_0, intersecting Q_0 at B, P at C. Denote $u := \frac{\overrightarrow{OC}}{|\overrightarrow{OC}|}$, then $R_\theta = \mathrm{span}\{u, v\}$, $\Phi_\theta(v) = u$ and $\Phi_\theta(u) = -v$.

Denote

$$r(P) := \sum_{\theta \in \mathrm{Arg}(P,Q_0) \cap (0,\pi/2]} m_\theta = \sum_{\theta \in \mathrm{Arg}(P^\perp, Q_0^\perp) \cap (0,\pi/2]} m_\theta^\perp \tag{1.12}$$

then $0 \in \mathrm{Arg}(P, Q_0)$ if and only if $r(P) < m$, and $m_0 = m - r(P)$. Similarly $0 \in \mathrm{Arg}(P^\perp, Q_0^\perp)$ if and only if $r(P) < n$, and $m_0^\perp = n - r(P)$.

1.2. **Angle space distributions and submanifolds with CJA.** Let M be an n-dimensional submanifold in $\mathbb{R}^{n+m}$ and Q_0 be a fixed m-plane in $\mathbb{R}^{n+m}$. Denote by TM and NM the tangent bundle and the normal bundle along M, respectively.

For any $p \in M$, denote by $\mathrm{Arg}(N_pM, Q_0)$ ($\mathrm{Arg}(T_pM, Q_0^\perp)$) the set consisting of all the Jordan angles between N_pM (T_pM) and Q_0 ($Q_0^\perp$), which are called *normal (tangent) Jordan angles* at p. Let θ be a $[0, \pi/2]$-valued smooth function on M, if $\theta(p) \in \mathrm{Arg}(N_pM, Q_0)$ ($\theta(p) \in \mathrm{Arg}(T_pM, Q_0^\perp)$), we say θ is a *normal (tangent) Jordan angle function* of M relative to Q_0. Denote by Arg^N (Arg^T) the set consisting of all the normal (tangent) Jordan angle functions of M relative to Q_0. If θ is a smooth function on M that is nonzero everywhere, then Lemma 1.1 implies $\theta \in \mathrm{Arg}^N$ if and only if $\theta \in \mathrm{Arg}^T$.

Denote

$$\begin{aligned} N_\theta M &:= \{\nu \in N_pM : p \in M, \nu \text{ is an angle direction associated to } \theta(p)\}, \\ T_\theta M &:= \{v \in T_pM : p \in M, v \text{ is an angle direction associated to } \theta(p)\}. \end{aligned} \tag{1.13}$$

Let $\mathcal{P}_0$ and $\mathcal{P}_0^\perp$ be orthogonal projections onto Q_0 and $Q_0^\perp$, $(\cdot)^T$ and $(\cdot)^N$ denote orthogonal projections onto T_pM and N_pM, respectively. Then $\nu \in N_{p,\theta}M := N_\theta M \cap N_pM$ if and only if

$$(\mathcal{P}_0 \nu)^N = \cos^2\theta(p)\nu \tag{1.14}$$

and similarly $u \in T_{p,\theta}M := T_\theta M \cap T_pM$ if and only if

$$(\mathcal{P}_0^\perp u)^T = \cos^2\theta(p)u. \tag{1.15}$$

Let $m_\theta^N(p) := \dim N_{p,\theta}M$, $m_\theta^T(p) := \dim T_{p,\theta}M$ for every $p \in M$, then m_θ^N and m_θ^T are both $\mathbb{Z}^+$-valued functions on M.

Based on [29], one can easily deduced that

Lemma 1.2. *([24]) Let θ be a normal (tangent) Jordan angle function of M relative to Q_0. If m_θ^N (m_θ^T) is a constant function on M, then $N_\theta M$ ($T_\theta M$) is a smooth subbundle of NM (TM).*

In this case, $N_\theta M$ ($T_\theta M$) is said to be a *normal (tangent) angle space distribution* associated to θ. A curve $\gamma : t \in (a,b) \mapsto \gamma(t) \in M$, all of whose tangent vectors belongs to a tangent angle space distribution, i.e. $\dot\gamma(t) \in T_\theta M$ for every $t \in (a,b)$, is called an *angle line* of M. More generally, an *angle surface* is a connected submanifold S of M, such that for any $p \in S$, $T_pS \subset T_\theta M$.

Now we can formulate the definition of *submanifolds with constant Jordan angles (CJA)*, the main subject of this paper.

Definition 1.1. *Let M be an n-dimensional submanifold of $\mathbb{R}^{n+m}$ and Q_0 be a fixed m-plane. If every normal Jordan angle function of M relative to Q_0 is a constant function, and m_θ^N is constant on M for each $\theta \in Arg^N$, then we say M has* ***constant Jordan angles (CJA)*** *relative to Q_0.*

With the aid of Lemma 1.1 and Proposition 1.1, one can obtain equivalent definitions of submanifolds with CJA.

Proposition 1.2. *For any n-dimensional submanifold M of $\mathbb{R}^{n+m}$ and a fixed m-plane Q_0, the following statement are equivalent:*

(i) M has CJA relative to Q_0;

(ii) Every tangent angle function of M relative to $Q_0^\perp$ is constant, and m_θ^T is constant;

(iii) Arg(N_pM, Q_0) (Arg($T_pM, Q_0^\perp$)) is independent of $p \in M$, and the multiplicity of each normal (tangent) Jordan angle is constant;

(iv) The relative position of N_pM (T_pM) and Q_0 ($Q_0^\perp$) is independent of $p \in M$.

Remarks:

- Let γ be an arc-length parameterized curve in $\mathbb{R}^3$. If γ is a *constant angle curve*, i.e. the unit tangent vector at every point makes a constant angle with a fixed straight line in $\mathbb{R}^3$, then γ is a helix, and vise versa. Let S be a smooth surface in $\mathbb{R}^3$, if the normal vector at every point makes a constant angle with a fixed straight line in $\mathbb{R}^3$, then S is said to be a *constant angle surface* in $\mathbb{R}^3$. A surface S in $\mathbb{R}^3$ is a constant angle surface if and only if it is locally isometric to either a cylinder, a right circular cone, or the tangential developable of a helix. Moreover, if we additionally assume S to be complete, then S has to be a cylinder. Recently, many geometers are interested in constant angle surfaces in other ambient spaces, e.g. $S^2 \times \mathbb{R}$ [10], $\mathbb{H}^2 \times \mathbb{R}$ [12], Heisenberg group [14], Minkowski space [27] and product spaces [11]. Our notion is a natural generalization of the classical constant angle curves and surfaces.
- If M^n is a hypersurface of $\mathbb{R}^{n+1}$, then M has CJA if and only if M is a helix hypersurface [9]. Hence the concept of submanifolds with CJA is a natural generalization of helix hypersurfaces to higher codimensional cases. Helix hypersurfaces are closed related to the shadow problem (see [17]) formulated by H. Wente, and another interesting motivation for the study of helix hypersurfaces comes from the physics of interfaces of liquid crystal (see [6]).
- Let S be a surface in $\mathbb{R}^4$, then S has CJA if and only if S is a surface in $\mathbb{R}^4$ with *constant principal angles with respect to a plane*. This concept was introduced by Bayard-Di Scala-Castro-Hernández in [3]. In this paper, the authors established a local existence theorem and classified all the complete surfaces in $\mathbb{R}^4$ with constant principal angles.

Denote

$$r := \sum_{\theta \in \mathrm{Arg}^N, \theta \neq 0} m_\theta^N, \tag{1.16}$$

then r is a constant $\mathbb{Z}^+$-valued function on M. As shown above, $0 \in \mathrm{Arg}^N$ ($0 \in \mathrm{Arg}^T$) if and only if $r < m$ ($r < n$), and the multiplicity of 0 equals $m - r$ ($n - r$). Let

$$g^N := |\mathrm{Arg}^N| \qquad g^T := |\mathrm{Arg}^T| \tag{1.17}$$

be the numbers of distinct Jordan angles. Note that $g^N = g^T + 1$ whenever $r \equiv n < m$, $g^T = g^N + 1$ whenever $r \equiv m < n$, and otherwise $g^N = g^T$.

In conjunction with Lemma 1.1 and Lemma 1.2, NM and TM have the following vector bundle decompositions

$$\begin{aligned} NM &= \bigoplus_{\theta\in \mathrm{Arg}^N} N_\theta M, \\ TM &= \bigoplus_{\theta\in \mathrm{Arg}^T} T_\theta M. \end{aligned} \tag{1.18}$$

In particular, if $\theta\neq 0,\pi/2$, then there exists a smooth mapping $\Phi_\theta : R_\theta M\to R_\theta M$, where

$$R_\theta M := N_\theta M\oplus T_\theta M, \tag{1.19}$$

such that: (i) Φ_θ keeps each fiber invariant; (ii) the length of each vector in $R_\theta M$ is invariant under Φ_θ; (iii) $\Phi_\theta^2 = -\mathbf{Id}$; (iv) $\Phi_\theta(N_\theta M) = T_\theta M$, $\Phi_\theta(T_\theta M) = N_\theta M$; (iv) for any $\nu\in N_\theta M$ and $u\in T_\theta M$,

$$\begin{aligned} \sec\theta\ \mathcal{P}_0\nu &= \cos\theta\ \nu - \sin\theta\ \Phi_\theta(\nu), \\ \sec\theta\ \mathcal{P}_0^\perp u &= \cos\theta\ u - \sin\theta\ \Phi_\theta(u). \end{aligned} \tag{1.20}$$

Φ_θ is called the *anti-involution* associated to θ.

1.3. **Minimal submanifolds with CJA and the Bernstein problem.** The concept of CJA submanifolds that we have just introduced arises from our systematic investigation of the Bernstein problem in higher codimension. We now wish to explain this connection.

The classical Bernstein theorem [4] states that any entire minimal graph in $\mathbb{R}^3$ has to be affine linear. This result has been extended by J. Simons [30] to such entire minimal graphs in $\mathbb{R}^{n+1}$ for $n\leq 7$, whereas Bombieri-de Giorgi-Giusti [5] constructed counterexamples in higher dimensions. But for any dimensions, there is a weak version of the Bernstein type theorem, obtained by J. Moser [28] who proved that any entire solution $f:\mathbb{R}^n\to\mathbb{R}$ to the minimal surface equation

$$\mathrm{div}\Big(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\Big) = 0 \tag{1.21}$$

has to be affine linear, provided that

$$v := \sqrt{1+|\nabla f|^2} \tag{1.22}$$

is a bounded function. v is a significant quantity here for various reasons. Firstly, the boundedness of v ensures that (1.21) is a uniformly elliptic equation, so that a Bernstein type result can be obtained by Moser's iteration. Secondly, for any $f:\mathbb{R}^n\to\mathbb{R}$, $x=(x^1,\cdots,x^n)\in\mathbb{R}^n\mapsto (x,f(x))\in \mathrm{graph}\ f$ is a global coordinate chart of the graph of f, and a straightforward calculation shows that the volume form of graph f is $v dx^1\wedge\cdots\wedge dx^n$, i.e. v equals the radio of the volume form of graph f and the coordinate plane. Thirdly, v has a close relationship with Jordan angles. A direct computation shows

$$\nu := w(-\frac{\partial f}{\partial x^1},\cdots,-\frac{\partial f}{\partial x^n},1) \qquad \text{where } w:=v^{-1} \tag{1.23}$$

is a unit normal vector field on graph f. Thus the angle between ν and the x^{n+1}-axis is $\arccos w$, which is smaller than an acute angle whenever the v-function is bounded. Therefore, Moser's theorem can be restated as: Let M be a complete minimal hypersurface in $\mathbb{R}^{n+1}$ and $\theta_0 \in (0, \pi/2)$. If the angle between the normal vector and x^{n+1}-axis is smaller than θ_0 everywhere, then M has to be an affine n-plane.

Now we consider an n-dimensional entire minimal graph M in $\mathbb{R}^{n+m}$, generated by a smooth vector-valued function $f : \mathbb{R}^n \to \mathbb{R}^m$

$$x = (x^1, \cdots, x^n) \mapsto f(x) = (f^1(x), \cdots, f^m(x)).$$

Then f satisfies the minimal surface equations

$$\begin{aligned} \sum_i \frac{\partial}{\partial x^i}(vg^{ij}) &= 0 \qquad \forall j = 1, \cdots, n, \\ \sum_{i,j} \frac{\partial}{\partial x^i}(vg^{ij}\frac{\partial f^\alpha}{\partial x^j}) &= 0 \qquad \forall \alpha = 1, \cdots, m. \end{aligned} \tag{1.24}$$

Here $g_{ij}dx^i dx^j$ is the induced metric on M, (g^{ij}) denotes the inverse matrix of (g_{ij}), and $v dx^1 \wedge \cdots \wedge dx^n := \det(g_{ij})^{1/2} dx^1 \wedge \cdots \wedge dx^n$ is the volume form of M. More precisely,

$$v = \Big[\det\big(\delta_{ij} + \sum_\alpha \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\alpha}{\partial x^j}\big)\Big]^{1/2}. \tag{1.25}$$

Similarly to the case of codimension 1, the v-function has a close relationship with Jordan angles. At any point $p \in M$, denote by

$$0 \le \theta_1 \le \theta_2 \le \cdots \le \theta_m < \pi/2 \tag{1.26}$$

the Jordan angles between N_pM and the coordinate m-plane, then a calculation shows (see [34][22])

$$v = \prod_{i=1}^m \sec\theta_m. \tag{1.27}$$

We note that

$$w := v^{-1} = \prod_{i=1}^m \cos\theta_m \tag{1.28}$$

is the inner product of the normal m-plane and the coordinate m-plane. Here all the m-planes are viewed as vectors in a Euclidean space of larger dimension, via Plücker embedding (see [23]).

It is natural to ask whether Moser's theorem can be generalized to the higher codimensional case. In other words, given an entire minimal graph $M = \text{graph } f \subset \mathbb{R}^{n+m}$ with $f : \mathbb{R}^n \to \mathbb{R}^m$, does the boundedness of the v-function ensure that M has to be an affine n-plane? The answer is 'Yes' for the cases of dimension 2 [8][25] and dimension 3 [2][15], but it is 'No' for dimension 4, according to the works of Lawson-Osserman [26] and Harvey-Lawson [18].

Let us explain the geometric reason for this fact. Let $\mathbb{O}$ and $\mathbb{H}$ denote the octonions and quaternions, respectively. We have $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}e$, with e a unit element orthogonal to $\mathbb{H}$, and for any $a, b, c, d \in \mathbb{H}$,

$$(a + be)(c + de) = (ac - \bar{d}b) + (da + b\bar{c})e. \tag{1.29}$$

Denote $\mathrm{Sp}_1 := \{q \in \mathbb{H} : |q| = 1\}$. Assume $a \in \mathrm{Im}\ \mathbb{H}$ is a fixed unit element, then

$$M(a) := \{r\big[(\sqrt{5}/2)qa\bar{q} + \bar{q}e\big] : q \in \mathrm{Sp}_1, r \in \mathbb{R}^+\} \tag{1.30}$$

is a 4-dimensional cone in $\mathrm{Im}\ \mathbb{O}$, which is the graph of the function $\eta : \mathbb{H}\backslash\{0\} \to \mathrm{Im}\ \mathbb{H}\backslash\{0\}$

$$\eta(x) = \frac{\sqrt{5}}{2|x|}\bar{x}\varepsilon x. \tag{1.31}$$

Here $\varepsilon \in \mathrm{Im}\ \mathbb{H}$ and $|\varepsilon| = 1$. Note that η is a *cone-like function*, i.e. $\eta(tx) = t\eta(x)$ for any t and x. It was discovered by Lawson-Osserman [26] that η is a Lipschitz solution to the non-parametric minimal surface equations that is not C^1, and a straightforward calculation shows the v-function is always 9 on $M(a)$. Afterwards, Harvey-Lawson [18] constructed a family of 4-dimensional entire minimal graphs in $\mathrm{Im}\ \mathbb{O}$; the tangent cone at infinity of each one is just the Lawson-Osserman's cone, and the v-function takes value in $[1, 9)$. Therefore, Moser's theorem cannot generalize to all higher codimensional cases.

Now we further explore the geometric properties of Lawson-Osserman's cone via Jordan angles.

Proposition 1.3. *Lawson-Osserman's cone $M(a)$ is a 4-dimensional submanifold in Im $\mathbb{O}$ with CJA relative to Im $\mathbb{H}$, and $Arg^N = \{\arccos(2/3), \arccos(\sqrt{6}/6)\}$, $Arg^T = \{\arccos(2/3), \arccos(\sqrt{6}/6), 0\}$.*

Remark. Proposition 1.3 was firstly proved in the Appendix of [23], and the calculation was based on the complex form of the Hopf map from S^3 to S^2. Now, we shall give another proof, which is based on the fact that $M(a)$ is a Sp_1-invariant manifold and has a close relationship with the argument in Section 3.

Proof. Denote $F : \mathrm{Sp}_1 \times \mathbb{R}^+ \to M(a)$

$$(q, r) \mapsto r\big[(\sqrt{5}/2)qa\bar{q} + \bar{q}e\big]. \tag{1.32}$$

Let $p_0 = F(q_0, R_0)$ be an arbitrary point in $M(a)$. We shall compute the Jordan angles between $T_{p_0}M(a)$ and $\mathbb{H}e$.

Let sp_1 be the Lie algebra associated to Sp_1, which can be seen as the linear space constisting of right-invariant vector fields on Sp_1. It is well-known that sp_1 is isomorphic to $\mathrm{Im}\ \mathbb{H}$, and the isomorphism is given by $\chi : \mathrm{Im}\ \mathbb{H} \to \mathrm{sp}_1$

$$b \mapsto V = \chi(b) \text{ with } V_q = \frac{d}{dt}\Big|_{t=0} e^{tb}q. \tag{1.33}$$

As a matter of convenience, b and $\chi(b)$ are regarded to be same in the sequel. Then at p_0,

$$F_*\partial_r = (\sqrt{5}/2)q_0a\bar{q}_0 + \bar{q}_0e = (\sqrt{5}/2)a_1 + \varepsilon$$

with

$$a_1 := q_0 a \bar{q}_0, \quad \varepsilon := \bar{q}_0 e$$

and

$$\begin{aligned} F_* b &= \frac{d}{dt}\Big|_{t=0} R_0\big[(\sqrt{5}/2)(e^{tb}q_0)a(\overline{e^{tb}q_0}) + (\overline{e^{tb}q_0})e)\big] \\ &= \frac{d}{dt}\Big|_{t=0} R_0\big[(\sqrt{5}/2)e^{tb}(q_0 a\bar{q}_0)e^{-tb} + (\bar{q}_0 e^{-tb})e\big] \\ &= \frac{d}{dt}\Big|_{t=0} R_0\big[(\sqrt{5}/2)e^{tb}a_1 e^{-tb} + e^{-tb}\varepsilon\big] \\ &= R_0\big[(\sqrt{5}/2)(ba_1 - a_1 b) - b\varepsilon\big]. \end{aligned}$$

Let a_2 be a unit vector in $\mathrm{Im}\,\mathbb{H}$ that is orthogonal to a_1 and denote $a_3 := a_1 a_2$. Then $\{a_1, a_2, a_3\}$ is an orthonormal basis of $\mathrm{Im}\,\mathbb{H}$, satisfying $a_1^2 = a_2^2 = a_3^2 = -1$, $a_1 a_2 = a_3 = -a_2 a_1$, $a_2 a_3 = a_1 = -a_3 a_2$ and $a_3 a_1 = a_2 = -a_1 a_3$, then

$$\begin{aligned} R_0^{-1} F_* a_1 &= (\sqrt{5}/2)(a_1^2 - a_1^2) - a_1\varepsilon = -a_1\varepsilon, \\ R_0^{-1} F_* a_2 &= (\sqrt{5}/2)(a_2 a_1 - a_1 a_2) - a_2\varepsilon = -\sqrt{5}a_3 - a_2\varepsilon, \\ R_0^{-1} F_* a_3 &= (\sqrt{5}/2)(a_3 a_1 - a_1 a_3) - a_3\varepsilon = \sqrt{5}a_2 - a_3\varepsilon. \end{aligned}$$

Denote

$$\begin{aligned} e_1 :=& (2/3)F_*\partial_r = (\sqrt{5}/3)a_1 + (2/3)\varepsilon, \\ e_2 :=& (\sqrt{6}R_0)^{-1}F_* a_2 = -(\sqrt{30}/6)a_3 - (\sqrt{6}/6)a_2\varepsilon, \\ e_3 :=& (\sqrt{6}R_0)^{-1}F_* a_3 = (\sqrt{30}/6)a_2 - (\sqrt{6}/6)a_3\varepsilon, \\ e_4 :=& R_0^{-1}F_* a_1 = -a_1\varepsilon. \end{aligned} \tag{1.34}$$

Then $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of $T_{p_0}M(a)$.

Let $\mathcal{P}_0$, $\mathcal{P}_0^\perp$ be the orthogonal projections of $\mathrm{Im}\,\mathbb{O} = \mathrm{Im}\,\mathbb{H} \oplus \mathbb{H}e$ into $\mathrm{Im}\,\mathbb{H}$ and $\mathbb{H}e$, respectively, then

$$\langle(\mathcal{P}_0^\perp e_1)^T, e_j\rangle = \langle\mathcal{P}_0^\perp e_1, e_j\rangle = \langle\mathcal{P}_0^\perp e_1, \mathcal{P}_0^\perp e_j\rangle = (4/9)\delta_{1j}$$

which implies $(\mathcal{P}_0^\perp e_1)^T = (4/9)e_1$ and hence e_1 is a tangent angle direction associated to $\theta_1 := \arccos(2/3)$. Note that e_1 is the direction of the ray going through p_0. Similarly, one can prove that e_2, e_3 are both tangent angle directions associated to $\theta := \arccos(\sqrt{6}/6)$, and e_4 is a tangent angle direction associated to 0. Since p_0 can be taken arbitrarily, $M(a)$ has CJA relative to $\mathrm{Im}\,\mathbb{H}$, and $\mathrm{Arg}^T = \{\theta_1, \theta, 0\}$, $\mathrm{Arg}^N = \{\theta_1, \theta\}$. Moreover, an arbitrary angle line with respect to θ_1 is a ray of $M(a)$, and vise versa. $\square$

In [26], Lawson-Osserman raised the following question: What is the largest constant C such that an entire minimal graph of arbitrary dimension and codimension with $v \le C$ has to be affine linear? Up to now, the best positive answer to this question in a successive series of achievements by several mathematicians (see [19], [21], [34], [22]) is gotten in [24], which says that for any entire minimal graph $M = \{(x, f(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+m}$ with $f : \mathbb{R}^n \to \mathbb{R}^m$, if $v \le 3$, then M has to be

an affine n-plane. But there is still a large quantitative gap between 3 and 9, that is, between known Bernstein type theorems and the counterexamples.

Lawson-Osserman's problem can be viewed as the first gap problem of the v-function for entire minimal graphs of higher codimension. To study the gap phenomena of the v-function, it is natural to consider minimal graphs whose v-function is constant. Observing that the v-function is a function of all Jordan angle functions (see (1.27)), the v-function on any minimal graph with CJA relative to the coordinate plane is constant. Proposition 1.3 shows the Lawson-Osserman's cone $M(a)$ has CJA relative to the imaginary quaternions, but unfortunately it is not a complete submanifold. So one can propose the following problems:

Problem 1.1. *Do there exist nonflat entire minimal graphs whose v-function is constant?*

Problem 1.2. *Let S_v and S_v^0 be sets consisting of some real numbers strictly bigger than 1. $v_0 \in S_v$ if and only if there exists a nonlinear cone-like map $f : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}^m\backslash\{0\}$, such that $M =$ graph f is a minimal graph whose v-function always equals v_0. Similarly, $v_0 \in S_v^0$ if and only if there exists a nonlinear cone-like map $f : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}^m\backslash\{0\}$, such that $M =$ graph f is a minimal graph with CJA relative to $\mathbb{R}^m$. Are S_v and S_v^0 discrete sets?*

Problem 1.3. *Let $S_{v,loc}$ and $S_{v,loc}^0$ be sets consisting of some real numbers strictly bigger than 1. $v_0 \in S_{v,loc}$ if and only if there exists a nonlinear vector-valued function $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ (D is an open domain), such that $M =$ graph f is a minimal graph whose v-function always equals v_0. Similarly, $v_0 \in S_{v,loc}^0$ if and only if there exists a nonlinear vector-valued function $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$, such that $M =$ graph f is a minimal graph with CJA relative to $\mathbb{R}^m$. Are $S_{v,loc}$ and $S_{v,loc}^0$ discrete sets?*

Problem 1.4. *Does any minimal graph in Euclidean space with constant v-function have to be a submanifold with CJA?*

Obviously $S_v^0 \subset S_v$, $S_{v,loc}^0 \subset S_{v,loc}$, $S_v \subset S_{v,loc}$, $S_v^0 \subset S_{v,loc}^0$ and Problem 1.3 can be viewed as a local version of Problem 1.2. For Problem 1.2, the known facts include $(1, 3] \notin S_v$ (see [24]) and $9 \in S_v^0$.

Problem 1.2 is quite similar to Chern's conjecture, intrinsic rigidity problem in the theory of minimal submanifolds, which claims that if the squared length of the second fundamental form (denoted by $|B|^2$) of a compact minimal submanifold in the unit Euclidean sphere is constant, then the value should be contained in a discrete set (see [7]).

1.4. **Submanifolds in spheres with CJA.** If M is an n-dimensional cone in $\mathbb{R}^{n+m}$, then the intersection of M and the unit sphere gives an $(n-1)$-dimensional submanifold N in S^{n+m-1}. M is said to be the cone generated by N, i.e. $M = CN$. As pointed out by J. Simons [30], the geometric properties of N are closed related to those of the cone CN. Firstly, CN has parallel mean curvature in $\mathbb{R}^{n+m}$ if and only if N is a minimal submanifold in S^{n+m-1} (see [32] p.64). Noting that CN is a linear subspace if and only if N is a totally geodesic subsphere, the Bernstein problem

for minimal submanifolds in Euclidean space can be transferred to the spherical Bernstein problem for minimal submanifolds in the sphere, in the framework of the geometric measure theory (see [1], [16]).

For any $p \in N$, denote by T_pN and N_pN the tangent $(n-1)$-plane and the normal m-plane of N at p, respectively, then

$$T_pS^{n+m-1} = T_pN \oplus N_pN.$$

Along the ray going through p, the tangent n-planes and the normal m-planes of CN are both constant, and

$$T_p(CN) = T_pN \oplus \{t\mathbf{X}(p) : t \in \mathbb{R}\}, \quad N_p(CN) = N_pN. \tag{1.35}$$

Here $\mathbf{X}(p)$ denotes the position vector of p in $\mathbb{R}^{n+m}$.

Let Q_0 be a fixed m-plane in $\mathbb{R}^{n+m}$, if $\mathrm{Arg}(N_pN, Q_0)$ is independent of $p \in N$, and the multiplicity of each normal Jordan angle is constant, then we say N is a **submanifold in a sphere with constant Jordan angles (CJA)** relative to Q_0. By (1.35), N has CJA if and only if the cone CN generated by N is a submanifold in $\mathbb{R}^{n+m}$ with CJA.

Thereby, Problem 1.3 can be restated as follows:

Problem 1.5. *Let S_w and S_w^0 be sets consisting of some real numbers taking values in $(0,1)$. $w_0 \in S_w$ if and only if there exists an $(n-1)$-dimensional compact minimal submanifold N in S^{n+m-1}, that is non-totally geodesic, such that its w-function always equals w_0, i.e. the inner product of each normal m-plane and a fixed m-plane Q_0 is w_0. Similarly, $w_0 \in S_w^0$ if and only if there exists an $(n-1)$-dimensional compact minimal and non-totally geodesic submanifold N in S^{n+m-1}, which has CJA relative to a fixed m-plane, such that its w-function always equals w_0. Are S_w and S_w^0 discrete?*

Remark. Due to (1.28),

$$S_w = \{w_0 = v_0^{-1} : v_0 \in S_v\}, \quad S_w^0 = \{w_0 = v_0^{-1} : v_0 \in S_v^0\}.$$

There is a long way to resolving these problems. In this paper, we only consider CJA submanifolds with a small number of distinct Jordan angles (i.e. g^N and g^T).

1.5. **Main results.** This paper will be organized as follows.

In Section 2, the second fundamental form B of submanifolds with CJA in Euclidean space shall be studied. At first, differentiating the Jordan angle functions not only gives some nullity properties of B, but also reveals the relationship between the induced tangent (normal) connection and the second fundamental form. Taking the covariant derivative of the formulas obtained in the previous step, one can compute some components of ∇B in terms of B. With the aid of the Codazzi equations, we can derive a constraint equation for the second fundamental form (see Lemma 2.6), which is nontrivial when the multiplicity of a tangent Jordan angle function $\theta \in (0, \pi/2)$, i.e. m_θ^T, is strictly larger than 1. This conclusion will play

an important part in Section 3. Based on these formulas, it is easy to get some vanishing theorems for the second fundamental form B of submanifolds with CJA, including the following one.

Theorem 1.1. *Let f be an $\mathbb{R}^m$-valued function on an open domain $D \subset \mathbb{R}^n$. If $M = \text{graph } f$ is a minimal submanifold with CJA relative to $\mathbb{R}^m$, and $g^N, g^T \leq 2$, then f has to be affine linear, i.e. M has to be an affine n-plane.*

Note that the example of Lawson-Osserman's cone implies that the condition '$g^N, g^T \leq 2$' in Theorem 1.1 cannot be omitted.

In [18], Harvey-Lawson introduced a new concept of coassociative submanifolds, as an important example of calibrated geometries, and showed that Lawson-Osserman's cone is a coassociative submanifold. Observing that coassociative submanifolds constitute an important class of 4-dimensional minimal submanifolds in $\mathbb{R}^7$, it is natural to study the structure of coassociative submanifolds with CJA, which is the main topic of Section 3. With the aid of the algebraic properties of octonions, one can obtain several interesting conclusions on the Jordan angles and the second fundamental form of coassociative submanifolds. In conjunction with Lemma 2.6, a structure theorem for coassociative submanifolds with CJA is deduced as follows.

Theorem 1.2. *Let f be a smooth function from an open domain $D \subset \mathbb{H}$ into $Im\, \mathbb{H}$. If $M = \text{graph } f$ is a coassociative submanifold with CJA relative to $Im\, \mathbb{H}$, and $g^N \leq 2, g^T \leq 3$, then f is either an affine linear function or $f(x) = \eta(x - x_0) + y_0$, where $x_0 \in \mathbb{H}$, $y_0 \in Im\, \mathbb{H}$ and*

$$\eta(x) = \frac{\sqrt{5}}{2|x|}\bar{x}\varepsilon x$$

with ε an arbitrary unit element in $Im\, \mathbb{H}$. In other words, M is an affine 4-plane or a translate of an open subset of the Lawson-Osserman's cone.

2. On the second fundamental form of submanifolds with CJA

Let M be an n-dimensional submanifold in $\mathbb{R}^{n+m}$ with CJA relative to a fixed m-plane Q_0. We use the notations $\mathcal{P}_0$, $\mathcal{P}_0^{\perp}$, Arg^N, Arg^T, $N_\theta M$, $T_\theta M$, $R_\theta M$, m_θ^N, m_θ^T, g^N, g^T established in Section 1. For p in M, we put

$$N_{p,\theta}M := N_pM \cap N_\theta M, \qquad T_{p,\theta}M := T_pM \cap T_\theta M. \tag{2.1}$$

The second fundamental form B is a pointwise symmetric bilinear form on T_pM ($p \in M$) with values in N_pM defined by

$$B_{XY} = (\overline{\nabla}_X Y)^N$$

with $\overline{\nabla}$ the Levi-Civita connection on $\mathbb{R}^{n+m}$. The induced connections on TM and NM are

$$\nabla_X Y = (\overline{\nabla}_X Y)^T, \qquad \nabla_X \nu = (\overline{\nabla}_X \nu)^N.$$

Here X, Y are smooth sections of TM and ν denotes a smooth section of NM. The second fundamental form, the curvature tensor of the submanifold, the curvature tensor of the normal bundle and the curvature tensor of the ambient manifold satisfy the Gauss, Codazzi and Ricci equations (see [33] for details).

Let A be the shape operator defined by

$$A^\nu(v) = (-\overline{\nabla}_v \nu)^T \qquad \forall \nu \in \Gamma(NM), v \in T_pM. \tag{2.2}$$

A^ν is a symmetric operator on T_pM and satisfies the Weingarten equations

$$\langle B_{XY}, \nu\rangle = \langle A^\nu(X), Y\rangle \qquad \forall X, Y \in \Gamma(TM). \tag{2.3}$$

The trace of the second fundamental form gives a normal vector field H on M, which is called the mean curvature vector field. If $\nabla H \equiv 0$, then we say that M has parallel mean curvature. Moreover if $H \equiv 0$, M is called a minimal submanifold.

2.1. **Nullity lemmas.** Let $\theta \in \mathrm{Arg}^N(\theta \neq 0, \pi/2)$ and $\Phi_\theta : R_\theta M \to R_\theta M$ denote the anti-involution associated to θ, then (1.20) gives

$$\begin{aligned}\mathcal{P}_0^\perp v &= \cos\theta\big(\cos\theta\ v - \sin\theta\ \Phi_\theta(v)\big)\\ &= \cos^2\theta\ v - \cos\theta\sin\theta\ \Phi_\theta(v)\end{aligned}$$

for any $v \in T_\theta M$ and

$$\begin{aligned}\mathcal{P}_0\mu &= \cos\theta\big(\cos\theta\ \mu - \sin\theta\ \Phi_\theta(\mu)\big)\\ &= \cos^2\theta\ \mu - \cos\theta\sin\theta\ \Phi_\theta(\mu)\end{aligned}$$

for any $\mu \in N_\theta M$. In other words,

$$\begin{aligned}(\mathcal{P}_0^\perp v)^T &= \cos^2\theta\ v, \qquad (\mathcal{P}_0^\perp v)^N = -\cos\theta\sin\theta\ \Phi_\theta(v),\\ (\mathcal{P}_0\mu)^N &= \cos^2\theta\ \mu, \qquad (\mathcal{P}_0\mu)^T = -\cos\theta\sin\theta\ \Phi_\theta(\mu).\end{aligned} \tag{2.4}$$

Based on the above formulas, one can easily deduce the following nullity lemmas for the second fundamental form of M.

Lemma 2.1. *For each $\theta \in Arg^N$ which takes values in $(0, \pi/2)$,*

$$\langle B_{uv}, \Phi_\theta(w)\rangle + \langle B_{uw}, \Phi_\theta(v)\rangle = 0 \tag{2.5}$$

holds pointwisely for any $u \in T_pM$ and $v, w \in T_{p,\theta}M$. In particular,

$$\langle B_{uv}, \Phi_\theta(v)\rangle = 0 \tag{2.6}$$

for every $v \in T_{p,\theta}M$.

Proof. By linearity, it suffices to prove (2.6) for any unit vector $v \in T_{p,\theta}M$.

Let X be a smooth local section of $T_\theta M$, such that $X_p = v$ and $|X| \equiv 1$, then

$$\langle \mathcal{P}_0^\perp X, \mathcal{P}_0^\perp X\rangle = |\mathcal{P}_0^\perp X|^2 \equiv \cos^2\theta. \tag{2.7}$$

Differentiating both sides with respect to u yields

$$\begin{aligned}
0 &= (1/2)\nabla_u\langle \mathcal{P}_0^\perp X, \mathcal{P}_0^\perp X\rangle = \langle \overline{\nabla}_u(\mathcal{P}_0^\perp X), \mathcal{P}_0^\perp v\rangle \\
&= \langle \mathcal{P}_0^\perp(\overline{\nabla}_u X), \mathcal{P}_0^\perp v\rangle = \langle \mathcal{P}_0^\perp(\nabla_u X), \mathcal{P}_0^\perp v\rangle + \langle \mathcal{P}_0^\perp B_{uv}, \mathcal{P}_0^\perp v\rangle \\
&= \langle \nabla_u X, (\mathcal{P}_0^\perp v)^T\rangle + \langle B_{uv}, (\mathcal{P}_0^\perp v)^N\rangle \\
&= \cos^2\theta\langle \nabla_u X, v\rangle - \cos\theta\sin\theta\langle B_{uv}, \Phi_\theta(v)\rangle \\
&= (1/2)\cos^2\theta\nabla_u|X|^2 - \cos\theta\sin\theta\langle B_{uv}, \Phi_\theta(v)\rangle \\
&= -\cos\theta\sin\theta\langle B_{uv}, \Phi_\theta(v)\rangle
\end{aligned}$$

(where we have used (2.4)) and then we arrive at (2.6).

$\square$

Lemma 2.2. *For each $\theta \in Arg^N$ taking values in $(0, \pi/2)$,*

$$\langle B_{uv}, \nu\rangle = 0 \tag{2.8}$$

for any $u, v \in T_{p,\theta}M$ and $\nu \in N_{p,\theta}M$.

Proof. Let $w := -\Phi_\theta(\nu)$, then $w \in T_{p,\theta}M$ and $\Phi_\theta(w) = -\Phi_\theta^2(\nu) = \nu$. Applying Lemma 2.1 gives

$$\begin{aligned}
\langle B_{uv}, \nu\rangle &= \langle B_{uv}, \Phi_\theta(w)\rangle = -\langle B_{uw}, \Phi_\theta(v)\rangle \\
&= -\langle B_{wu}, \Phi_\theta(v)\rangle = \langle B_{wv}, \Phi_\theta(u)\rangle \\
&= \langle B_{vw}, \Phi_\theta(u)\rangle = -\langle B_{vu}, \Phi_\theta(w)\rangle \\
&= -\langle B_{uv}, \Phi_\theta(w)\rangle = -\langle B_{uv}, \nu\rangle
\end{aligned}$$

and (2.8) immediately follows from the above equation.

$\square$

Lemma 2.3. *If $\theta \in Arg^N \cap Arg^T$ and $\theta \equiv 0$ or $\pi/2$, then*

$$\langle B_{uv}, \nu\rangle = 0 \tag{2.9}$$

for any $u \in T_pM$, $v \in T_{p,\theta}M$ and $\nu \in N_{p,\theta}M$.

Proof. If $\theta \equiv 0$, let X be a smooth local section of $T_\theta M$ such that $X_p = v$, then $X_q \in Q_0^\perp$ for any q. Thus $(\overline{\nabla}_u X)_p \subset Q_0^\perp$. On the other hand, $\nu \in N_{p,\theta}M$ implies $\nu \in Q_0$, hence

$$\langle B_{uv}, \nu\rangle = \langle \overline{\nabla}_u X, \nu\rangle = 0.$$

The proof for $\theta \equiv \pi/2$ is similar. $\square$

2.2. **Connections.** Let $\theta,\sigma\in \mathrm{Arg}^T$, $\theta\neq\sigma$, X a local section of TM, Y and Z local sections of $T_\theta M$ and $T_\sigma M$, respectively. Define

$$(S_{\theta\sigma})_{YZ}(X):=\langle\nabla_X Y,Z\rangle. \tag{2.10}$$

Then for any smooth function f defined on M, $(S_{\theta\sigma})_{YZ}(fX)=f(S_{\theta\sigma})_{YZ}(X)$, $(S_{\theta\sigma})_{Y,fZ}(X)=f(S_{\theta\sigma})_{YZ}(X)$ and

$$\begin{aligned}(S_{\theta\sigma})_{fY,Z}(X)&=\langle\nabla_X(fY),Z\rangle=f\langle\nabla_X Y,Z\rangle+(\nabla_X f)\langle Y,Z\rangle\\&=f(S_{\theta\sigma})_{YZ}(X).\end{aligned}$$

This means $S_{\theta\sigma}$ is a smooth tensor field on M of type $(3,0)$. More precisely, $S_{\theta\sigma}$ is a smooth section of the tensor bundle $T^*M\otimes T_\theta^*M\otimes T_\sigma^*M$. Since ∇ is a Levi-Civita connection on M,

$$\begin{aligned}(S_{\theta\sigma})_{YZ}(X)&=\langle\nabla_X Y,Z\rangle=\nabla_X\langle Y,Z\rangle-\langle\nabla_X Z,Y\rangle\\&=-\langle\nabla_X Z,Y\rangle=-(S_{\sigma\theta})_{ZY}(X).\end{aligned} \tag{2.11}$$

Now we additionally define

$$\Phi_\theta|_{R_\theta M}=0\qquad\text{whenever }\theta\equiv 0\text{ or }\pi/2, \tag{2.12}$$

then (2.4) still holds when $\theta=0$ or $\pi/2$. Let

$$\kappa_{\theta\sigma}:=\frac{\sin 2\theta}{\cos 2\theta-\cos 2\sigma} \tag{2.13}$$

be a constant depending only on θ and σ. The following result reveals the relationship between $S_{\theta\sigma}$ and the second fundamental form.

Lemma 2.4. *Let $\theta,\sigma\in Arg^T$, $\theta\neq\sigma$, then for any $u\in T_pM$, $v\in T_{p,\theta}M$ and $w\in T_{p,\sigma}M$,*

$$(S_{\theta\sigma})_{vw}(u)=\kappa_{\sigma\theta}\langle B_{uv},\Phi_\sigma(w)\rangle-\kappa_{\theta\sigma}\langle B_{uw},\Phi_\theta(v)\rangle. \tag{2.14}$$

Proof. Let Y,Z be smooth local sections of $T_\theta M$ and $T_\sigma M$, respectively, such that $Y(p)=v$, $Z(p)=w$, then $(\mathcal{P}_0^\perp Y)^T=\cos^2\theta\ Y$, $(\mathcal{P}_0^\perp Z)^T=\cos^2\sigma\ Z$. Hence

$$\begin{aligned}0=\cos^2\theta\langle Y,Z\rangle&=\langle(\mathcal{P}_0^\perp Y)^T,Z\rangle\\&=\langle\mathcal{P}_0^\perp Y,Z\rangle=\langle\mathcal{P}_0^\perp Y,\mathcal{P}_0^\perp Z\rangle.\end{aligned}$$

Differentiating both sides of the above equation with respect to $u\in T_pM$ yields

$$\begin{aligned}0&=\nabla_u\langle\mathcal{P}_0^\perp Y,\mathcal{P}_0^\perp Z\rangle=\langle\overline{\nabla}_u(\mathcal{P}_0^\perp Y),\mathcal{P}_0^\perp w\rangle+\langle\mathcal{P}_0^\perp v,\overline{\nabla}_u(\mathcal{P}_0^\perp Z)\rangle\\&=\langle\mathcal{P}_0^\perp(\overline{\nabla}_u Y),\mathcal{P}_0^\perp w\rangle+\langle\mathcal{P}_0^\perp v,\mathcal{P}_0^\perp(\overline{\nabla}_u Z)\rangle\\&=\langle\mathcal{P}_0^\perp(\nabla_u Y),\mathcal{P}_0^\perp w\rangle+\langle\mathcal{P}_0^\perp v,\mathcal{P}_0^\perp(\nabla_u Z)\rangle+\langle\mathcal{P}_0^\perp B_{uv},\mathcal{P}_0^\perp w\rangle+\langle\mathcal{P}_0^\perp v,\mathcal{P}_0^\perp B_{uw}\rangle\\&=\langle\nabla_u Y,(\mathcal{P}_0^\perp w)^T\rangle+\langle\nabla_u Z,(\mathcal{P}_0^\perp v)^T\rangle+\langle B_{uv},(\mathcal{P}_0^\perp w)^N\rangle+\langle B_{uw},(\mathcal{P}_0^\perp v)^N\rangle\\&=\cos^2\sigma\langle\nabla_u Y,w\rangle+\cos^2\theta\langle\nabla_u Z,v\rangle-\cos\sigma\sin\sigma\langle B_{uv},\Phi_\sigma(w)\rangle-\cos\theta\sin\theta\langle B_{uw},\Phi_\theta(v)\rangle\\&=(\cos^2\sigma-\cos^2\theta)(S_{\theta\sigma})_{vw}(u)-\cos\sigma\sin\sigma\langle B_{uv},\Phi_\sigma(w)\rangle-\cos\theta\sin\theta\langle B_{uw},\Phi_\theta(v)\rangle\\&=(1/2)(\cos 2\sigma-\cos 2\theta)(S_{\theta\sigma})_{vw}(u)-(1/2)\sin 2\sigma\langle B_{uv},\Phi_\sigma(w)\rangle-(1/2)\sin 2\theta\langle B_{uw},\Phi_\theta(v)\rangle\end{aligned}$$

(we have used (2.4) and (2.11)), which is equivalent to (2.14).

□

Similarly, given $u \in T_pM$, $\mu \in \Gamma(N_\theta M)$, $\nu \in \Gamma(N_\sigma M)$ with $\theta, \sigma \in \mathrm{Arg}^N$ and $\theta \neq \sigma$, one can define

$$(S^N_{\theta\sigma})_{\mu\nu}(u) := \langle \nabla_u \mu, \nu \rangle. \tag{2.15}$$

Then $S^N_{\theta\sigma}$ is a smooth section of $T^*M \otimes N^*_\theta M \otimes N^*_\sigma M$, and

$$\begin{aligned}(S^N_{\sigma\theta})_{\nu\mu}(u) &= \langle \nabla_u \nu, \mu\rangle = \nabla_u \langle \nu, \mu\rangle - \langle \nu, \nabla_u \mu\rangle \\ &= -\langle \nabla_u \mu, \nu\rangle = -(S^N_{\theta\sigma})_{\mu\nu}(u).\end{aligned} \tag{2.16}$$

Let μ, ν be local section of $N_\theta M$ and $N_\sigma M$ respectively, then

$$\begin{aligned}0 = \cos^2\theta \langle \mu, \nu\rangle &= \langle (\mathcal{P}_0 \mu)^N, \nu\rangle \\ &= \langle \mathcal{P}_0\mu, \nu\rangle = \langle \mathcal{P}_0\mu, \mathcal{P}_0\nu\rangle.\end{aligned} \tag{2.17}$$

Differentiating both sides of the above equality with respect to $u \in T_pM$, one can use (2.4) to get the following result, as in the proof of Lemma 2.4.

Lemma 2.5. *Given $\theta, \sigma \in Arg^N$, $\theta \neq \sigma$,*

$$(S^N_{\theta\sigma})_{\mu\nu}(u) = \kappa_{\theta\sigma} \langle B_{u, \Phi_\theta(\mu)}, \nu\rangle - \kappa_{\sigma\theta}\langle B_{u,\Phi_\sigma(\nu)}, \mu\rangle \tag{2.18}$$

for any $u \in T_pM, \mu \in N_{p,\theta}M$ and $\nu \in N_{p,\sigma}M$.

2.3. **Computation of ∇B and related results.** Let $\theta \in \mathrm{Arg}^T$, $\sigma \in \mathrm{Arg}^N$, and $(\cdot)^\sigma$ be the orthogonal projection of N_pM onto $N_{p,\sigma}M$. Define

$$R_{\theta\sigma}(v_1, v_2, v_3, v_4) := \langle B^\sigma_{v_1v_3}, B^\sigma_{v_2v_4}\rangle - \langle B^\sigma_{v_1v_4}, B^\sigma_{v_2v_3}\rangle \tag{2.19}$$

for any $v_1, v_2, v_3, v_4 \in T_{p,\theta}M$. Then $R_{\theta\sigma}$ is a smooth section of the tensor bundle $T^*_\theta M \otimes T^*_\theta M \otimes T^*_\theta M \otimes T^*_\theta M$. Obviously $R_{\theta\sigma}(v_1, v_2, v_3, v_4) = -R_{\theta\sigma}(v_2, v_1, v_3, v_4) = -R_{\theta\sigma}(v_1, v_2, v_4, v_3) = R_{\theta\sigma}(v_3, v_4, v_1, v_2)$, and

$$\begin{aligned}&R_{\theta\sigma}(v_1, v_2, v_3, v_4) + R_{\theta\sigma}(v_2, v_3, v_1, v_4) + R_{\theta\sigma}(v_3, v_1, v_2, v_4) \\ =&\langle B^\sigma_{v_1v_3}, B^\sigma_{v_2v_4}\rangle - \langle B^\sigma_{v_1v_4}, B^\sigma_{v_2v_3}\rangle + \langle B^\sigma_{v_2v_1}, B^\sigma_{v_3v_4}\rangle \\ &- \langle B^\sigma_{v_2v_4}, B^\sigma_{v_3v_1}\rangle + \langle B^\sigma_{v_3v_2}, B^\sigma_{v_1v_4}\rangle - \langle B^\sigma_{v_3v_4}, B^\sigma_{v_1v_2}\rangle \\ =&0.\end{aligned} \tag{2.20}$$

Hence $R_{\theta\sigma}$ is a curvature type tensor. Note that $R_{\theta\sigma} = 0$ whenever $m^T_\theta \equiv 1$.

Let $\theta, \sigma \in \mathrm{Arg}^T$, and define

$$U_{\theta\sigma}(v_1, v_2, v_3, v_4) =: \left\langle (A^{\Phi_\theta(v_3)} v_1)_\sigma, (A^{\Phi_\theta(v_4)} v_2)_\sigma \right\rangle - \left\langle (A^{\Phi_\theta(v_4)} v_1)_\sigma, (A^{\Phi_\theta(v_3)} v_2)_\sigma \right\rangle \tag{2.21}$$

for any $v_1, v_2, v_3, v_4 \in T_{p,\theta}M$. Here $(\cdot)_\sigma$ denotes the orthogonal projection of T_pM onto $T_{p,\sigma}M$. Due to Lemma 2.1, $A^{\Phi_\theta(v)} w + A^{\Phi_\theta(w)} v = 0$ for any $v, w \in T_{p,\theta}M$, hence $U_{\theta\sigma}(v_1, v_2, v_3, v_4) = -U_{\theta\sigma}(v_2, v_1, v_3, v_4) = -U_{\theta\sigma}(v_1, v_2, v_4, v_3) = U_{\theta\sigma}(v_3, v_4, v_1, v_2)$ and

$U_{\theta\sigma} = 0$ whenever $m_\theta^T \equiv 1$. Note, however, that $U_{\theta\sigma}$ does not satisfy a Bianchi type identity.

Lemma 2.6. *Given $\theta \in Arg^T$ taking values in $(0, \pi/2)$,*

$$\sum_{\sigma \in Arg^N, \sigma \neq \theta} \kappa_{\theta\sigma} R_{\theta\sigma}(v, w, v, w) = 3 \sum_{\sigma \in Arg^T, \sigma \neq \theta} \kappa_{\theta\sigma} U_{\theta\sigma}(v, w, v, w) \tag{2.22}$$

for any $v, w \in T_{p,\theta}M$, and moreover

$$\begin{aligned} \langle (\nabla_v B)_{ww}, \Phi_\theta(v)\rangle =& (1/3) \sum_{\sigma \in Arg^N, \sigma \neq \theta} \kappa_{\theta\sigma} \left(\langle B^\sigma_{vv}, B^\sigma_{ww}\rangle + 2|B^\sigma_{vw}|^2\right) \\ & - 2 \sum_{\sigma \in Arg^T, \sigma \neq \theta} \kappa_{\sigma\theta} \langle B^\sigma_{vw}, \Phi_\sigma(A^{\Phi_\theta(v)} w)_\sigma\rangle. \end{aligned} \tag{2.23}$$

Proof. Let

$$u_\sigma := (A^{\Phi_\theta(v)} w)_\sigma \tag{2.24}$$

for each $\sigma \in \mathrm{Arg}^T$, then Lemma 2.1 tells us

$$(A^{\Phi_\theta(w)} v)_\sigma = -(A^{\Phi_\theta(v)} w)_\sigma = -u_\sigma \tag{2.25}$$

and moreover

$$\begin{aligned} U_{\theta\sigma}(v, w, v, w) &= \left\langle (A^{\Phi_\theta(v)} v)_\sigma, (A^{\Phi_\theta(w)} w)_\sigma \right\rangle - \left\langle (A^{\Phi_\theta(w)} v)_\sigma, (A^{\Phi_\theta(v)} w)_\sigma \right\rangle \\ &= |u_\sigma|^2. \end{aligned} \tag{2.26}$$

In particular, combining the Weingarten equations and Lemma 2.2 gives

$$|u_\theta|^2 = \langle u_\theta, A^{\Phi_\theta(v)} w\rangle = \langle B_{u_\theta w}, \Phi_\theta(v)\rangle = 0,$$

i.e. $u_\theta = 0$.

Let Y, Z be local sections of $T_\theta M$ such that $Y_p = v$, $Z_p = w$. By Lemma 2.2, $\langle B_{ZZ}, \Phi_\theta(Y)\rangle \equiv 0$, hence

$$\begin{aligned} \langle (\nabla_v B)_{ww}, \Phi_\theta(v)\rangle &= \nabla_v \langle B_{ZZ}, \Phi_\theta(Y)\rangle - \langle B_{ww}, \nabla_v \Phi_\theta(Y)\rangle - 2\langle B_{\nabla_v Z, w}, \Phi_\theta(v)\rangle \\ &= -\langle B_{ww}, \nabla_v \Phi_\theta(Y)\rangle - 2\langle B_{\nabla_v Z, w}, \Phi_\theta(v)\rangle \\ &:= -I - 2II \end{aligned} \tag{2.27}$$

where

$$\begin{aligned} I = \sum_{\sigma \in \mathrm{Arg}^N} \langle B^\sigma_{ww}, \nabla_v \Phi_\theta(Y)\rangle &= \sum_{\sigma \in \mathrm{Arg}^N, \sigma \neq \theta} (S^N_{\theta\sigma})_{\Phi_\theta(v), B^\sigma_{ww}}(v) \\ &= \sum_{\sigma \in \mathrm{Arg}^N, \sigma \neq \theta} \Big(\kappa_{\theta\sigma} \langle B_{v, \Phi^2_\theta(v)}, B^\sigma_{ww}\rangle - \kappa_{\sigma\theta} \langle B_{v, \Phi_\sigma(B^\sigma_{ww})}, \Phi_\theta(v)\rangle\Big) \\ &= - \sum_{\sigma \in \mathrm{Arg}^N, \sigma \neq \theta} \kappa_{\theta\sigma} \langle B^\sigma_{vv}, B^\sigma_{ww}\rangle \end{aligned} \tag{2.28}$$

(Lemma 2.2, Lemma 2.1, Lemma 2.5 and $\Phi_\theta^2 = -\mathbf{Id}$ have been used in this calculation) and

$$\begin{aligned} II = \langle B_{\nabla_v Z, w}, \Phi_\theta(v)\rangle &= \sum_{\sigma\in \mathrm{Arg}^T} \langle \nabla_v Z, u_\sigma\rangle = \sum_{\sigma\in\mathrm{Arg}^T, \sigma\neq\theta} (S_{\theta\sigma})_{wu_\sigma}(v) \\ &= \sum_{\sigma\in\mathrm{Arg}^T, \sigma\neq\theta} (\kappa_{\sigma\theta}\langle B_{vw}, \Phi_\sigma(u_\sigma)\rangle - \kappa_{\theta\sigma}\langle B_{vu_\sigma}, \Phi_\theta(w)\rangle) \\ &= \sum_{\sigma\in\mathrm{Arg}^T, \sigma\neq\theta} \left(\kappa_{\sigma\theta}\langle B_{vw}, \Phi_\sigma(u_\sigma)\rangle + \kappa_{\theta\sigma}|u_\sigma|^2\right). \end{aligned} \tag{2.29}$$

(Here we have used the Weingarten equations, (2.25) and Lemma 2.4.) Substituting (2.28) and (2.29) into (2.27) implies

$$\langle(\nabla_v B)_{ww}, \Phi_\theta(v)\rangle = \sum_{\sigma\in\mathrm{Arg}^N, \sigma\neq\theta} \kappa_{\theta\sigma}\langle B^\sigma_{vv}, B^\sigma_{ww}\rangle - 2\sum_{\sigma\in\mathrm{Arg}^T, \sigma\neq\theta} \left(\kappa_{\sigma\theta}\langle B_{vw}, \Phi_\sigma(u_\sigma)\rangle + \kappa_{\theta\sigma}|u_\sigma|^2\right). \tag{2.30}$$

Again applying Lemma 2.2 gives $\langle B_{ZY}, \Phi_\theta(Y)\rangle \equiv 0$, hence

$$\begin{aligned} \langle(\nabla_w B)_{wv}, \Phi_\theta(v)\rangle =& \nabla_w\langle B_{ZY}, \Phi_\theta(Y)\rangle - \langle B_{wv}, \nabla_w\Phi_\theta(Y)\rangle \\ &- \langle B_{\nabla_w Z, v}, \Phi_\theta(v)\rangle - \langle B_{w,\nabla_w Y}, \Phi_\theta(v)\rangle \\ =& - \langle B_{wv}, \nabla_w\Phi_\theta(Y)\rangle - \langle B_{w,\nabla_w Y}, \Phi_\theta(v)\rangle \\ :=& - I - II \end{aligned} \tag{2.31}$$

where

$$\begin{aligned} I &= \sum_{\sigma\in\mathrm{Arg}^N} \langle B^\sigma_{wv}, \nabla_w\Phi_\theta(Y)\rangle = \sum_{\sigma\in\mathrm{Arg}^N, \sigma\neq\theta} (S^N_{\theta\sigma})_{\Phi_\theta(v), B^\sigma_{wv}}(w) \\ &= \sum_{\sigma\in\mathrm{Arg}^N, \sigma\neq\theta} \Big(\kappa_{\theta\sigma}\langle B_{w,\Phi_\theta^2(v)}, B^\sigma_{wv}\rangle - \kappa_{\sigma\theta}\langle B_{w,\Phi_\sigma(B^\sigma_{wv})}, \Phi_\theta(v)\rangle\Big) \\ &= \sum_{\sigma\in\mathrm{Arg}^N, \sigma\neq\theta} \left(-\kappa_{\theta\sigma}|B^\sigma_{wv}|^2 - \kappa_{\sigma\theta}\langle u_\sigma, \Phi_\sigma(B^\sigma_{wv})\rangle\right) \end{aligned} \tag{2.32}$$

and

$$\begin{aligned} II =& \langle B_{w,\nabla_w Y}, \Phi_\theta(v)\rangle = \sum_{\sigma\in\mathrm{Arg}^T} \langle \nabla_w Y, u_\sigma\rangle = \sum_{\sigma\in\mathrm{Arg}^T, \sigma\neq\theta} (S_{\theta\sigma})_{vu_\sigma}(w) \\ &= \sum_{\sigma\in\mathrm{Arg}^T, \sigma\neq\theta} (\kappa_{\sigma\theta}\langle B_{wv}, \Phi_\sigma(u_\sigma)\rangle - \kappa_{\theta\sigma}\langle B_{wu_\sigma}, \Phi_\theta(v)\rangle) \\ &= \sum_{\sigma\in\mathrm{Arg}^T, \sigma\neq\theta} \left(\kappa_{\sigma\theta}\langle B_{wv}, \Phi_\sigma(u_\sigma)\rangle - \kappa_{\theta\sigma}|u_\sigma|^2\right). \end{aligned} \tag{2.33}$$

If $\sigma \neq 0, \pi/2$, then Φ_σ is isometric and $\Phi_\sigma^2 = -\mathbf{Id}$. Hence

$$\langle u_\sigma, \Phi_\sigma(B^\sigma_{wv})\rangle = \langle \Phi_\sigma(u_\sigma), \Phi^2_\sigma(B^\sigma_{wv})\rangle = -\langle B^\sigma_{wv}, \Phi_\sigma(u_\sigma)\rangle.$$

On the other hand, $\Phi_\sigma = 0$ whenever $\sigma = 0$ or $\pi/2$. Therefore

$$-\sum_{\sigma\in\mathrm{Arg}^N, \sigma\neq\theta} \kappa_{\sigma\theta}\langle u_\sigma, \Phi_\sigma(B^\sigma_{wv})\rangle = \sum_{\sigma\in\mathrm{Arg}^T, \sigma\neq\theta} \kappa_{\sigma\theta}\langle B_{wv}, \Phi_\sigma(u_\sigma)\rangle. \tag{2.34}$$

Substituting (2.32)-(2.34) into (2.31) yields

$$\langle(\nabla_w B)_{wv},\Phi_\theta(v)\rangle=\sum_{\sigma\in\mathrm{Arg}^N,\sigma\neq\theta}\kappa_{\theta\sigma}|B^\sigma_{wv}|^2+\sum_{\sigma\in\mathrm{Arg}^T,\sigma\neq\theta}\left(\kappa_{\theta\sigma}|u_\sigma|^2-2\kappa_{\sigma\theta}\langle B_{wv},\Phi_\sigma(u_\sigma)\rangle\right). \tag{2.35}$$

The Codazzi equations imply $(\nabla_v B)_{ww}=(\nabla_w B)_{wv}$. Hence by comparing the right hand sides of (2.30) and (2.35) we arrive at

$$\sum_{\sigma\in\mathrm{Arg}^N,\sigma\neq\theta}\kappa_{\theta\sigma}\left(\langle B^\sigma_{vv},B^\sigma_{ww}\rangle-|B^\sigma_{vw}|^2\right)=3\sum_{\sigma\in\mathrm{Arg}^T,\sigma\neq\theta}\kappa_{\theta\sigma}|u_\sigma|^2 \tag{2.36}$$

and then (2.22) immediately follows from the definition of $R_{\theta\sigma}$ and $U_{\theta\sigma}$. Finally (2.23) is obtained by substituting (2.36) into (2.35).

□

Lemma 2.7. *We consider $\theta\in Arg^T$ taking values in $(0,\pi/2)$ and $\sigma\in Arg^T$ such that $\theta\neq\sigma$. If $U_{\theta\sigma}(v_1,v_2,v_1,v_2)=0$ holds for any $v_1,v_2\in T_{p,\theta}M$, then*

$$\begin{aligned}\langle(\nabla_v B)_{ww},\Phi_\theta(v)\rangle=&-2\sum_{\tau\in Arg^T,\tau\neq\theta}\kappa_{\tau\theta}\langle B^\tau_{vw},\Phi_\tau(A^{\Phi_\theta(v)}w)_\tau\rangle\\&+\sum_{\tau\in Arg^N,\tau\neq\theta}\kappa_{\theta\tau}|B^\tau_{vw}|^2+\sum_{\tau\in Arg^T,\tau\neq\theta}\kappa_{\theta\tau}|(A^{\Phi_\theta(v)}w)_\tau|^2\end{aligned} \tag{2.37}$$

for any $v\in T_{p,\theta}M$ and $w\in T_{p,\sigma}M$.

Proof. In the sequel we make use of the abbreviation $u_\tau:=(A^{\Phi_\theta(v)}w)_\tau$ for any $\tau\in\mathrm{Arg}^T$. By the definition of $U_{\theta\sigma}$,

$$\begin{aligned}0&=U_{\theta\sigma}(u_\theta,v,u_\theta,v)\\&=\left\langle(A^{\Phi_\theta(u_\theta)}u_\theta)_\sigma,(A^{\Phi_\theta(v)}v)_\sigma\right\rangle-\left\langle(A^{\Phi_\theta(v)}u_\theta)_\sigma,(A^{\Phi_\theta(u_\theta)}v)_\sigma\right\rangle\\&=|(A^{\Phi_\theta(v)}u_\theta)_\sigma|^2\end{aligned}$$

i.e. $(A^{\Phi_\theta(v)}u_\theta)_\sigma=0$. Hence

$$0=\langle A^{\Phi_\theta(v)}u_\theta,w\rangle=\langle A^{\Phi_\theta(v)}w,u_\theta\rangle=|u_\theta|^2$$

i.e. $u_\theta=0$. Similarly, one can deduce that $B^\theta_{vw}=0$.

Let Y be a local smooth section of $T_\theta M$ and Z be a local smooth section of $T_\sigma M$, such that $Y_p=v$, $Z_p=w$. Lemma 2.1 implies $\langle B_{YZ},\Phi_\theta(Y)\rangle\equiv 0$, hence

$$\begin{aligned}\langle(\nabla_v B)_{ww},\Phi_\theta(v)\rangle=&\langle(\nabla_w B)_{vw},\Phi_\theta(v)\rangle\\=&\langle\nabla_w\langle B_{YZ},\Phi_\theta(Y)\rangle-\langle B_{vw},\nabla_w\Phi_\theta(Y)\rangle\\&-\langle B_{\nabla_w Y,w},\Phi_\theta(v)\rangle-\langle B_{v,\nabla_w Z},\Phi_\theta(v)\rangle\\=&-\langle B_{vw},\nabla_w\Phi_\theta(Y)\rangle-\langle B_{\nabla_w Y,w},\Phi_\theta(v)\rangle\\:=&-I-II\end{aligned} \tag{2.38}$$

where

$$
\begin{aligned}
I = & \sum_{\tau\in \mathrm{Arg}^N} \langle B^\tau_{vw}, \nabla_w \Phi_\theta(Y)\rangle = \sum_{\tau\in \mathrm{Arg}^N, \tau\neq\theta} (S^N_{\theta\tau})_{\Phi_\theta(v), B^\tau_{vw}}(w) \\
= & \sum_{\tau\in \mathrm{Arg}^N, \tau\neq\theta} \Big(\kappa_{\theta\tau}\langle B_{w,\Phi^2_\theta(v)}, B^\tau_{vw}\rangle - \kappa_{\tau\theta}\langle B_{w,\Phi_\tau(B^\tau_{vw})}, \Phi_\theta(v)\rangle\Big) \\
= & - \sum_{\tau\in \mathrm{Arg}^N, \tau\neq\theta} \big(\kappa_{\theta\tau}|B^\tau_{vw}|^2 + \kappa_{\tau\theta}\langle \Phi_\tau(B^\tau_{vw}), u_\tau\rangle\big) \\
= & \sum_{\tau\in \mathrm{Arg}^N, \tau\neq\theta} \kappa_{\tau\theta}\langle B^\tau_{vw}, \Phi_\tau(u_\tau)\rangle - \sum_{\tau\in \mathrm{Arg}^N, \tau\neq\theta} \kappa_{\theta\tau}|B^\tau_{vw}|^2
\end{aligned} \tag{2.39}
$$

and

$$
\begin{aligned}
II = & \sum_{\tau\in \mathrm{Arg}^T} \langle \nabla_w Y, u_\tau\rangle = \sum_{\tau\in \mathrm{Arg}^T, \tau\neq\theta} (S_{\theta\tau})_{v,u_\tau}(w) \\
= & \sum_{\tau\in \mathrm{Arg}^T, \tau\neq\theta} \big(\kappa_{\tau\theta}\langle B_{vw}, \Phi_\tau(u_\tau)\rangle - \kappa_{\theta\tau}\langle B_{wu_\tau}, \Phi_\theta(v)\rangle\big) \\
= & \sum_{\tau\in \mathrm{Arg}^T, \tau\neq\theta} \kappa_{\tau\theta}\langle B^\tau_{vw}, \Phi_\tau(u_\tau)\rangle - \sum_{\tau\in \mathrm{Arg}^T, \tau\neq\theta} \kappa_{\theta\tau}|u_\tau|^2.
\end{aligned} \tag{2.40}
$$

Substituting (2.39) and (2.40) into (2.38) yields (2.37). □

Lemma 2.8. *If $\theta \in Arg^T \cap Arg^N$ and $\theta \equiv 0$ or $\pi/2$, then for any $v \in T_{p,\theta}M$, $\nu \in \Gamma(N_\theta M)$ and $w \in T_pM$,*

$$
\langle (\nabla_v B)_{ww}, \nu\rangle = -2 \sum_{\sigma\in Arg^T} \kappa_{\sigma\theta}\langle B^\sigma_{vw}, \Phi_\sigma(A^\nu w)_\sigma\rangle. \tag{2.41}
$$

Proof. Let Y be a local section of $T_\theta M$ and Z be a local section of TM, such that $Y_p = v$ and $Z_p = w$, then Lemma 2.3 tells us $\langle B_{YZ}, \nu\rangle \equiv 0$. Therefore

$$
\begin{aligned}
\langle (\nabla_v B)_{ww}, \nu\rangle = & \langle (\nabla_w B)_{vw}, \nu\rangle \\
= & \nabla_w \langle B_{YZ}, \nu\rangle - \langle B_{vw}, \nabla_w \nu\rangle - \langle B_{\nabla_w Y, w}, \nu\rangle - \langle B_{v,\nabla_w Z}, \nu\rangle \\
= & - \langle B_{vw}, \nabla_w \nu\rangle - \langle B_{\nabla_w Y, w}, \nu\rangle \\
:= & - I - II
\end{aligned} \tag{2.42}
$$

where

$$
\begin{aligned}
I = & \sum_{\sigma\in \mathrm{Arg}^N, \sigma\neq\theta} \langle B^\sigma_{vw}, \nabla_w \nu\rangle = \sum_{\sigma\in \mathrm{Arg}^N, \sigma\neq\theta} (S^N_{\theta\sigma})_{\nu, B^\sigma_{vw}}(w) \\
= & - \sum_{\sigma\in \mathrm{Arg}^N, \sigma\neq\theta} \kappa_{\sigma\theta}\langle B_{w,\Phi_\sigma(B^\sigma_{vw})}, \nu\rangle = - \sum_{\sigma\in \mathrm{Arg}^N} \kappa_{\sigma\theta}\langle \Phi_\sigma(B^\sigma_{vw}), u_\sigma\rangle \\
= & \sum_{\sigma\in \mathrm{Arg}^T} \langle B^\sigma_{vw}, \Phi_\sigma(u_\sigma)\rangle
\end{aligned} \tag{2.43}
$$

and

$$
\begin{aligned}
II &= \sum_{\sigma\in \mathrm{Arg}^T,\sigma\neq\theta}\langle \nabla_w Y, u_\sigma\rangle = \sum_{\sigma\in \mathrm{Arg}^T,\sigma\neq\theta}(S_{\theta\sigma})_{vu_\sigma}(w)\\
&= \sum_{\sigma\in \mathrm{Arg}^T,\sigma\neq\theta}\kappa_{\sigma\theta}\langle B_{wv},\Phi_\sigma(u_\sigma)\rangle = \sum_{\sigma\in \mathrm{Arg}^T}\kappa_{\sigma\theta}\langle B^\sigma_{vw},\Phi_\sigma(u_\sigma)\rangle.
\end{aligned} \tag{2.44}
$$

Here $u_\sigma := (A^\nu w)_\sigma$, and $u_\theta = B^\theta_{vw} = 0$ is a direct corollary of Lemma 2.3. Substituting (2.43) and (2.44) into (2.42), we arrive at at (2.41).

□

2.4. **Vanishing theorems.** With the above lemmas, we can now derive vanishing theorems for the second fundamental form of submanifolds with CJA.

Theorem 2.1. *Let M^n be a submanifold of $\mathbb{R}^{n+m}$ with CJA relative to a fixed m-plane Q_0 (M need not be complete), then*

(i) If $g^T = g^N = 1$, then M has to be an affine linear subspace;

(ii) If $g^T = 1, g^N = 2, \pi/2 \notin Arg^T$ and M has parallel mean curvature, then M is affine linear;

(iii) If $g^T = 2, g^N = 1, \pi/2 \notin Arg^N$, and M has parallel mean curvature, then M is affine linear;

(iv) If $g^T = g^N = 2$, $Arg^N \neq \{0, \pi/2\}$, and M is minimal, then M is affine linear.

Remarks:

- Let $S^1 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1, x_3 = 0\}$ be a circle whose tangent vectors are all orthogonal to the x_3-axis, then S^1 has CJA and $\mathrm{Arg}^T = \{\pi/2\}$, $\mathrm{Arg}^N = \{\pi/2, 0\}$. It is easy to check that S^1 has parallel mean curvature. Hence the condition '$\pi/2 \notin \mathrm{Arg}^T$' cannot be dropped in (ii).
- Let $S := S^1 \times \mathbb{R}$ be a circular cylinder, whose normal vectors are all orthogonal to the x_3-axis, then S has CJA and $\mathrm{Arg}^N = \{\pi/2\}$, $\mathrm{Arg}^T = \{\pi/2, 0\}$. Its mean curvature vector field is parallel along S. Hence the condition '$\pi/2 \notin \mathrm{Arg}^N$' cannot be dropped in (iii).
- Let S be a nontrivial minimal surface in $\mathbb{R}^3$, then $M := S \times \mathbb{R}$ is a minimal submanifold in $\mathbb{R}^3 \times \mathbb{R}^2 = \mathbb{R}^5$. Then M has CJA relative to $Q_0 := \mathbb{R}^2$, and $\mathrm{Arg}^N = \mathrm{Arg}^T = \{0, \pi/2\}$. Hence the condition '$\mathrm{Arg}^N \neq \{0, \pi/2\}$' cannot be dropped in (iv).

Proof. (i) Denote $g^T = g^N = \{\theta\}$, then Lemma 2.2 and 2.3 tell us

$$\langle B_{vw}, \nu\rangle = 0$$

for any $v, w \in T_{p,\theta}M = T_pM$ and $\nu \in N_{p,\theta}M = N_pM$. Hence M is totally geodesic.

(ii) As shown in Section 1, there exists $\theta_0 \neq 0, \pi/2$, such that $\mathrm{Arg}^T = \{\theta_0\}$, $\mathrm{Arg}^N = \{0, \theta_0\}$.

By Lemma 2.6,

$$\begin{aligned}
&\kappa_{\theta_0 0}\big(\langle B^0_{vv}, B^0_{ww}\rangle - |B^0_{vw}|^2\big) = \kappa_{\theta_0 0} R_{\theta_0 0}(v,w,v,w) \\
&= \sum_{\sigma\in \mathrm{Arg}^N, \sigma\neq\theta_0} \kappa_{\theta_0\sigma} R_{\theta_0\sigma}(v,w,v,w) = 3\sum_{\sigma\in \mathrm{Arg}^T, \sigma\neq\theta_0} \kappa_{\theta_0\sigma} U_{\theta_0\sigma}(v,w,v,w) \\
&= 0
\end{aligned} \tag{2.45}$$

for any $v, w \in T_{p,\theta_0}M = T_pM$. In conjunction with $\kappa_{\theta_0 0} = \frac{\sin 2\theta_0}{\cos 2\theta_0 - 1} \neq 0$, we have $\langle B^0_{vv}, B^0_{ww}\rangle = |B^0_{vw}|^2$. Substituting it into (2.23) implies

$$\begin{aligned}
\langle (\nabla_v B)_{ww}, \Phi_{\theta_0}(v)\rangle &= (1/3)\kappa_{\theta_0 0}\big(\langle B^0_{vv}, B^0_{ww}\rangle + 2|B^0_{vw}|^2\big) \\
&= \kappa_{\theta_0 0}|B^0_{vw}|^2.
\end{aligned} \tag{2.46}$$

Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of $T_{p,\theta_0}M = T_pM$. Since M has parallel mean curvature,

$$0 = \sum_{i=1}^n \langle \nabla_v H, \Phi_{\theta_0}(v)\rangle = \sum_{i=1}^n \langle (\nabla_v B)_{e_ie_i}, \Phi_{\theta_0}(v)\rangle = \kappa_{\theta_0 0}\sum_{i=1}^n |B^0_{ve_i}|^2 \tag{2.47}$$

which forces $|B^0_{ve_i}| = 0$ for any $1 \leq i \leq n$. Thus $B^0_{vw} = 0$ for any $v, w \in T_pM$. On the other hand, Lemma 2.2 implies $B^{\theta_0}_{vw} = 0$. Therefore $B \equiv 0$ on M.

(iii) Denote $\mathrm{Arg}^N = \{\theta_0\}$, $\mathrm{Arg}^T = \{0, \theta_0\}$ with $\theta_0 \neq 0, \pi/2$. Again applying Lemma 2.6 gives

$$\kappa_{\theta_0 0} U_{\theta_0 0}(v,w,v,w) = (1/3)\sum_{\sigma\in \mathrm{Arg}^N, \sigma\neq\theta_0} \kappa_{\theta_0\sigma} R_{\theta_0\sigma}(v,w,v,w) = 0 \tag{2.48}$$

i.e. $U_{\theta 0}(v,w,v,w) = 0$ for any $v, w \in T_{p,\theta_0}M$. This means

$$\begin{aligned}
0 &= \langle (A^{\Phi_{\theta_0}(v)}v)_0, (A^{\Phi_{\theta_0}(w)}w)_0\rangle - \langle (A^{\Phi_{\theta_0}(w)}v)_0, (A^{\Phi_{\theta_0}(v)}w)_0\rangle \\
&= \big|(A^{\Phi_{\theta_0}(v)}w)_0\big|^2.
\end{aligned} \tag{2.49}$$

Since $\Phi_{\theta_0} : T_{p,\theta_0}M \to N_{p,\theta_0}M = N_pM$ is an isomorphism, $(A^\nu w)_0 = 0$ holds for every $\nu \in N_pM$. On the other hand, $(A^\nu w)_{\theta_0} = 0$ is a direct corollary of Lemma 2.2. Thus $A^\nu w = 0$ for every $w \in T_{p,\theta_0}M$.

Let $\{e_1, \cdots, e_{m_{\theta_0}}\}$ be an orthonormal basis of $T_{p,\theta_0}M$, and $\{e_{m_{\theta_0}+1}, \cdots, e_n\}$ be an orthonormal basis of $T_{p,0}M$. For any $v \in T_{p,\theta_0}M$, by (2.23) and (2.37),

$$\langle (\nabla_v B)_{e_ie_i}, \Phi_{\theta_0}(v)\rangle = \begin{cases} 0 & \text{if } 1 \leq i \leq m_{\theta_0}, \\ \kappa_{\theta_0 0}\big|(A^{\Phi_{\theta_0}(v)}e_i)_0\big|^2 & \text{if } m_{\theta_0}+1 \leq i \leq n. \end{cases} \tag{2.50}$$

Hence

$$
\begin{aligned}
0 = \langle \nabla_v H, \Phi_{\theta_0}(v)\rangle &= \sum_{i=1}^{n} \langle (\nabla_v B)_{e_i e_i}, \Phi_{\theta_0}(v)\rangle \\
&= \sum_{i=m_{\theta_0}+1}^{n} \kappa_{\theta_0 0}\big|(A^{\Phi_{\theta_0}(v)} e_i)_0\big|^2
\end{aligned}
\tag{2.51}
$$

and then $(A^\nu e_i)_0 = 0$ for any $\nu \in N_pM$. On the other hand, $\langle A^\nu e_i, v\rangle = \langle A^\nu v, e_i\rangle = 0$ holds for any $v \in T_{p,\theta_0}M$. Therefore $A^\nu e_i = 0$ for each $m_{\theta_0} + 1 \le i \le n$.

In summary, $A^\nu \equiv 0$ for any smooth section ν of NM and then M has to be affine linear.

(iv) Denote $\mathrm{Arg}^N = \mathrm{Arg}^T = \{\theta_1, \theta_2\}$. Without loss of generality one can assume $\theta_1 \in (0, \pi/2)$. Let $\{e_1, \cdots, e_m\}$ be an orthonormal basis of $T_{p,\theta_1}M$ and $\{e_{m+1}, \cdots, e_n\}$ be an orthonormal basis of $T_{p,\theta_2}M$. By Lemma 2.6, for any $1 \le i, j \le m$,

$$
\begin{aligned}
&\langle B^{\theta_2}_{e_i e_i}, B^{\theta_2}_{e_j e_j}\rangle - |B^{\theta_2}_{e_i e_j}|^2 = R_{\theta_1\theta_2}(e_i, e_j, e_i, e_j) \\
=&3\ U_{\theta_1\theta_2}(e_i, e_j, e_i, e_j) = 3\big|(A^{\Phi_{\theta_1}(e_j)} e_i)\big|^2
\end{aligned}
$$

i.e.

$$
\langle B^{\theta_2}_{e_i e_i}, B^{\theta_2}_{e_j e_j}\rangle = 3\big|(A^{\Phi_{\theta_1}(e_j)} e_i)\big|^2 + |B^{\theta_2}_{e_i e_j}|^2. \tag{2.52}
$$

On the other hand, Lemma 2.2 and Lemma 2.3 tell us $B^{\theta_2}_{e_i e_j} = 0$ for every $m + 1 \le i, j \le n$. Since M is a minimal submanifold,

$$
\begin{aligned}
0 = |H^{\theta_2}|^2 &= \Big|\sum_{i=1}^{n} B^{\theta_2}_{e_i e_i}\Big|^2 \\
&= \Big|\sum_{i=1}^{m} B^{\theta_2}_{e_i e_i}\Big|^2 = \sum_{i,j=1}^{m} \langle B^{\theta_2}_{e_i e_i}, B^{\theta_2}_{e_j e_j}\rangle \\
&= \sum_{i,j=1}^{m} \big(3|(A^{\Phi_{\theta_1}(e_j)} e_i)_{\theta_2}|^2 + |B^{\theta_2}_{e_i e_j}|^2\big).
\end{aligned}
\tag{2.53}
$$

Hence $(A^{\Phi_{\theta_1}(e_j)} e_i)_{\theta_2} = B^{\theta_2}_{e_i e_j} = 0$ for all $1 \le i, j \le m$. In other words, $B^{\theta_2}_{v_1 v_2} = 0$ for any $v_1, v_2 \in T_{p,\theta_1}M$, and $B^{\theta_1}_{vw} = 0$ for any $v \in T_{p,\theta_1}M$ and $w \in T_{p,\theta_2}M$, which follows from the Weingarten equations.

If $\theta_2 \in (0, \pi/2)$, then similarly one can deduce that $B^{\theta_1}_{w_1 w_2} = 0$ for any $w_1, w_2 \in T_{p,\theta_2}M$ and $B^{\theta_2}_{vw} = 0$ for any $v \in T_{p,\theta_1}M$ and $w \in T_{p,\theta_2}M$. In conjunction with $B^{\theta_1}_{v_1 v_2} = 0$ for any $v_1, v_2 \in T_{p,\theta_1}M$ and $B^{\theta_2}_{w_1 w_2} = 0$ for any $w_1, w_2 \in T_{p,\theta_2}M$, we have $B \equiv 0$ on M and M has to be totally geodesic.

If $\theta_2 = 0$ or $\pi/2$, then (2.23) implies

$$
\langle (\nabla_v B)_{e_i e_i}, \Phi_{\theta_1}(v)\rangle = (1/3)\kappa_{\theta_1\theta_2}\big(\langle B^{\theta_2}_{e_i e_i}, B^{\theta_2}_{vv}\rangle + 2|B^{\theta_2}_{e_i v}|^2\big) = 0 \tag{2.54}
$$

for any $v \in T_{p,\theta_1}M$ and each $1 \le i \le m$. Since $U_{\theta_1\theta_2}(v_1, v_2, v_1, v_2) = 0$ for any $v_1, v_2 \in T_{p,\theta_1}M$, (2.37) tells us

$$\langle (\nabla_v B)_{e_ie_i}, \Phi_{\theta_1}(v)\rangle = \kappa_{\theta_1\theta_2}|B^{\theta_2}_{ve_i}|^2 + \kappa_{\theta_1\theta_2}|(A^{\Phi_{\theta_1}(v)}e_i)_{\theta_2}|^2. \tag{2.55}$$

for each $m+1 \le i \le n$. Thus

$$\begin{aligned} 0 = \langle \nabla_v H, \Phi_\theta(v)\rangle &= \sum_{i=1}^{n} \langle (\nabla_v B)_{e_ie_i}, \Phi_\theta(v)\rangle \\ &= \sum_{i=m+1}^{n} \kappa_{\theta_1\theta_2}\left(|B^{\theta_2}_{ve_i}|^2 + |(A^{\Phi_{\theta_1}(v)}e_i)_{\theta_2}|^2\right), \end{aligned} \tag{2.56}$$

which forces $B^{\theta_2}_{ve_i} = (A^{\Phi_{\theta_1}(v)}e_i)_{\theta_2} = 0$ for each $m+1 \le i \le n$. In other words, $B^{\theta_2}_{vw} = 0$ for any $v \in T_{p,\theta_1}M$ and $w \in T_{p,\theta_2}M$, and $B^{\theta_1}_{w_1w_2} = 0$ for any $w_1, w_2 \in T_{p,\theta_2}M$. Therefore $B \equiv 0$ on M and M has to be affine linear.

□

Let $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ be a smooth vector-valued function, then for any $p \in M :=$ graph f, any Jordan angle between N_pM and the coordinate m-plane takes values in $[0, \pi/2)$ (see [34]). Hence Theorem 2.1 implies:

Corollary 2.1. *Let D be an open domain of $\mathbb{R}^n$ and $f : D \to \mathbb{R}^m$. If $M =$ graph f is a minimal submanifold with CJA relative to the coordinate m-plane, and $g^N, g^T \le 2$, then M has to be an affine n-plane.*

This is the Theorem 1.1 mentioned in §1.5.

3. Coassociative submanifolds with CJA

3.1. **Associative subspace of Im $\mathbb{O}$.** Let $\mathbb{O}$ denote the octonions, which is an 8-dimensional normed algebra over $\mathbb{R}$ with multiplicative unit 1. More precisely, $\mathbb{O}$ is equipped with an inner product $\langle\cdot,\cdot\rangle$, whose associated norm $|\cdot|$ satisfies

$$|xy| = |x||y| \tag{3.1}$$

for any $x, y \in \mathbb{O}$. Denote by Re $\mathbb{O}$ the 1-dimensional subspace spanned by 1, and by Im $\mathbb{O}$ the orthogonal complement of Re $\mathbb{O}$. Then every $x \in \mathbb{O}$ has a unique decomposition

$$x = \text{Re } x + \text{Im } x$$

with Re $x \in$ Re $\mathbb{O}$, Im $x \in$ Im $\mathbb{O}$. The conjugation of x is defined by

$$\bar{x} = \text{Re } x - \text{Im } x. \tag{3.2}$$

For $w \in \mathbb{O}$, let R_w (L_w) denote the linear operator of right (left) multiplication by w, respectively. With the aid of (3.1) and (3.2), one can easily deduce the following fundamental formulas (see Appendix IV.A of [18]):

$$\langle R_wx, R_wy\rangle = \langle x, y\rangle|w|^2, \qquad \langle L_wx, L_wy\rangle = \langle x, y\rangle|w|^2, \tag{3.3}$$

$$\langle x, R_w y\rangle = \langle R_{\bar{w}} x, y\rangle, \qquad \langle x, L_w y\rangle = \langle L_{\bar{w}} x, y\rangle, \tag{3.4}$$

$$\bar{\bar{x}} = x,\ \overline{xy} = \bar{y}\bar{x}, \qquad x\bar{x} = |x|^2,\ \langle x, y\rangle = \text{Re}\ x\bar{y}. \tag{3.5}$$

Let P be a 3-dimensional real subspace of Im $\mathbb{O}$, if $A := \text{Re}\ \mathbb{O} \oplus P$ is a quarternion subalgebra of $\mathbb{O}$ (i.e. A is isomorphic to $\mathbb{H}$), then P is said to be *associative*.

Lemma 3.1. *Let P be an associative subspace of Im $\mathbb{O}$ and x, y be unit elements in P that are orthogonal to each other, then $\{x, y, z := xy\}$ is an orthonormal basis of P, and*

$$xy = -yx = z, \quad yz = -zy = x, \quad zx = -xz = y. \tag{3.6}$$

Conversely, if $\{x, y, z\}$ is an orthonormal basis of an associative subspace P, then $z = xy$ or $-xy$.

Proof. Since Re $\mathbb{O} \oplus P$ is a subalgebra of $\mathbb{O}$, $xy \in \text{Re}\ \mathbb{O} \oplus P$. By (3.2) and (3.5),

$$\text{Re}\ (xy) = -\text{Re}\ (x\bar{y}) = -\langle x, y\rangle = 0,$$

i.e. $xy \in P$. Applying (3.3) and (3.1) gives

$$\begin{aligned}
\langle xy, x\rangle &= \langle L_x y, L_x 1\rangle = \langle y, 1\rangle |x|^2 = 0,\\
\langle xy, y\rangle &= \langle R_y x, R_y 1\rangle = \langle x, 1\rangle |y|^2 = 0,\\
|xy| &= |x||y| = 1.
\end{aligned}$$

Hence $\{x, y, z := xy\}$ is an orthonormal basis of P.

Similarly, one can show yx is also a unit element in P orthogonal to span$\{x, y\}$, hence $yx = z$ or $-z$. If $yx = z$, then

$$\begin{aligned}
(x+y)(x-y) &= x^2 - y^2 + yx - xy\\
&= -\,x\bar{x} + y\bar{y} + z - z = -|x|^2 + |y|^2\\
&= 0.
\end{aligned} \tag{3.7}$$

On the other hand, since x and y are linearly independent, $x + y, x - y \neq 0$ and it follows from (3.1) that $|(x+y)(x-y)| = |x+y||x-y| \neq 0$, which contradicts (3.7). Hence $yx = -z$ and it follows that

$$\begin{aligned}
yz &= y(-yx) = -y^2 x\\
&= y\bar{y}x = |y|^2 x = x.
\end{aligned}$$

Similarly one can prove $zy = -x$ and $zx = -xz = y$.

Conversely, if $\{x, y, z\}$ is an orthonormal basis of P, then z and xy are both unit elements orthogonal to span$\{x, y\}$, which implies $z = xy$ or $-xy$.

$\square$

Lemma 3.2. *Let A be a quarternion subalgebra of $\mathbb{O}$, $\varepsilon \in A^\perp$ with $|\varepsilon| = 1$, then $A\varepsilon \perp A$, $\mathbb{O} = A \oplus A\varepsilon$ and*

$$(x + y\varepsilon)(v + w\varepsilon) = (xv - \bar{w}y) + (wx + y\bar{v})\varepsilon \tag{3.8}$$

for any $x, y, v, w \in A$.

Proof. The lemma immediately follows from Lemma A.8 in [18]. □

3.2. **Jordan angles between associative subspaces.** Now we explore the Jordan angles between an associative subspace P and $\text{Im}\,\mathbb{H}$.

Case I. $0 \in \text{Arg}(P, \text{Im}\,\mathbb{H})$ and $m_0 \geq 2$. This means there exist 2 unit elements $a, b \in P \cap \text{Im}\,\mathbb{H}$ that are orthogonal to each other, then it follows from Lemma 3.1 that $\{a, b, ab\}$ is an orthonormal basis of $P \cap \text{Im}\,\mathbb{H}$. Hence $P = \text{Im}\,\mathbb{H}$ and $\text{Arg}(P, \text{Im}\,\mathbb{H}) = \{0\}$.

Case II. $\pi/2 \in \text{Arg}(P, \text{Im}\,\mathbb{H})$ and $m_{\pi/2} \geq 2$. Then there exists 2 unit elements $ae, be \in P \cap (\text{Im}\,\mathbb{H})^\perp = P \cap \mathbb{H}e$ that are orthogonal to each other. By Lemma 3.1, $(ae)(be) = -\bar{b}a$ is a unit vector in P, and $-\bar{b}a \in \mathbb{H} \cap \text{Im}\,\mathbb{O} = \text{Im}\,\mathbb{H}$. Hence $\text{Arg}(P, \text{Im}\,\mathbb{H}) = \{0, \pi/2\}$, $m_0 = 1$, $m_{\pi/2} = 2$, and P is spanned by ae, be and $-\bar{b}a$, which are the angle directions of P relative to $\text{Im}\,\mathbb{H}$.

Case III. $m_0 \leq 1$ and $m_{\pi/2} \leq 1$. (Note that $m_0 = 0$ ($m_{\pi/2} = 0$) means $0 \notin \text{Arg}(P, \text{Im}\,\mathbb{H})$ ($\pi/2 \notin \text{Arg}(P, \text{Im}\,\mathbb{H})$), respectively.) Firstly, we claim $m_0 + m_{\pi/2} \leq 1$. If not, there exist unit elements $a \in P \cap \text{Im}\,\mathbb{H}$ and $be \in P \cap \mathbb{H}e$; by Lemma 3.1, P is spanned by a, be and $a(be) = (ba)e \in \mathbb{H}e$; hence $m_{\pi/2} = 2$, contradicting $m_{\pi/2} \leq 1$.

Hence there exist mutually orthogonal elements $x_1, x_2 \in P$ that are unit angle directions of P relative to $\text{Im}\,\mathbb{H}$ associated to $\theta_1, \theta_2 \in \text{Arg}(P, \text{Im}\,\mathbb{H}) \cap (0, \pi/2)$, respectively. More precisely,

$$(\mathcal{P} \circ \mathcal{P}_0)x_\alpha = \cos^2\theta_\alpha x_\alpha \qquad \forall \alpha = 1, 2. \tag{3.9}$$

Here $\mathcal{P}_0$ denotes the orthogonal projection of $\text{Im}\,\mathbb{O}$ onto $\text{Im}\,\mathbb{H}$ and $\mathcal{P}$ denotes the orthogonal projection of $\text{Im}\,\mathbb{O}$ onto P. As in Section 1, we denote by $\mathcal{P}_0^\perp$ the orthogonal projection of $\text{Im}\,\mathbb{O}$ onto $\mathbb{H}e = (\text{Im}\,\mathbb{H})^\perp$, then

$$\begin{aligned} x_\alpha &= \mathcal{P}_0 x_\alpha + \mathcal{P}_0^\perp x_\alpha \\ &= \cos\theta_\alpha a_\alpha + \sin\theta_\alpha y_\alpha \end{aligned} \tag{3.10}$$

with $a_\alpha := \sec\theta_\alpha\, \mathcal{P}_0 x_\alpha \in \text{Im}\,\mathbb{H}$ and $y_\alpha := \csc\theta_\alpha\, \mathcal{P}^\perp x_\alpha \in \mathbb{H}e$, satisfying $|a_\alpha| = |y_\alpha| = 1$ for each $\alpha = 1, 2$. Let ε be the unique element in $\mathbb{O}$ satisfying $y_1 = a_1\varepsilon$, then for every $c \in \mathbb{H}$,

$$\begin{aligned} \langle \varepsilon, c \rangle &= \langle L_{a_1}\varepsilon, L_{a_1}c \rangle = \langle a_1\varepsilon, a_1 c \rangle \\ &= \langle y_1, a_1 c \rangle = 0, \end{aligned}$$

which implies $\varepsilon \in \mathbb{H}e$. And $|\varepsilon| = 1$ directly follows from $y_1 = a_1\varepsilon$ and $|y_1| = |a_1| = 1$. Similarly, one can prove that there exists a unique $b \in \mathbb{H}$ which satisfies $y_2 = b\varepsilon$, and moreover $|b| = 1$.

Let $x_3 := x_1 x_2$, then Lemma 3.2 enables us to obtain

$$\begin{aligned} x_3 =&(\cos\theta_1 a_1 + \sin\theta_1 a_1\varepsilon)(\cos\theta_2 a_2 + \sin\theta_2 b\varepsilon) \\ =&(\cos\theta_1\cos\theta_2 a_1 a_2 - \sin\theta_1\sin\theta_2 \bar{b}a_1) \\ &+ (\cos\theta_1\sin\theta_2 b a_1 + \sin\theta_1\cos\theta_2 a_1\bar{a}_2)\varepsilon. \end{aligned} \tag{3.11}$$

By Lemma 3.1, $\{x_1, x_2, x_3\}$ is an orthonormal basis of P, thus for each $\alpha = 1, 2$,

$$\begin{aligned} 0 &= \cos^2\theta_\alpha\langle x_\alpha, x_3\rangle = \langle(\mathcal{P}\circ\mathcal{P}_0)x_\alpha, x_3\rangle \\ &= \langle\mathcal{P}_0 x_\alpha, x_3\rangle = \langle\mathcal{P}_0 x_\alpha, \mathcal{P}_0 x_3\rangle. \end{aligned} \tag{3.12}$$

When $\alpha = 1$, the above equation gives

$$\begin{aligned} 0 &= \langle\mathcal{P}_0 x_1, \mathcal{P}_0 x_3\rangle = \langle\cos\theta_1 a_1, \cos\theta_1\cos\theta_2 a_1 a_2 - \sin\theta_1\sin\theta_2 \bar{b} a_1\rangle \\ &= \cos^2\theta_1\cos\theta_2\langle a_1, a_1 a_2\rangle - \cos\theta_1\sin\theta_1\sin\theta_2\langle a_1, \bar{b}a_1\rangle \\ &= \cos^2\theta_1\cos\theta_2\langle 1, a_2\rangle - \cos\theta_1\sin\theta_1\sin\theta_2\langle 1, \bar{b}\rangle \\ &= -\cos\theta_1\sin\theta_1\sin\theta_2\langle\bar{b}, 1\rangle. \end{aligned}$$

In conjunction with $\theta_1, \theta_2 \in (0, \pi/2)$ we have $\langle\bar{b}, 1\rangle = 0$, therefore $b \in \mathrm{Im}\,\mathbb{H}$. Letting $\alpha = 2$ in (3.12) yields

$$\begin{aligned} 0 &= \langle\mathcal{P}_0 x_2, \mathcal{P}_0 x_3\rangle = \langle\cos\theta_2 a_2, \cos\theta_1\cos\theta_2 a_1 a_2 - \sin\theta_1\sin\theta_2 \bar{b} a_1\rangle \\ &= \cos\theta_1\cos^2\theta_2\langle a_2, a_1 a_2\rangle - \cos\theta_2\sin\theta_1\sin\theta_2\langle a_2, \bar{b}a_1\rangle \\ &= -\cos\theta_2\sin\theta_1\sin\theta_2\langle a_2, \bar{b}a_1\rangle \end{aligned}$$

and moreover

$$\begin{aligned} 0 &= \langle a_2, \bar{b}a_1\rangle = \langle a_2, R_{a_1}\bar{b}\rangle = \langle R_{\bar{a}_1} a_2, \bar{b}\rangle \\ &= \langle a_2\bar{a}_1, \bar{b}\rangle = \langle -a_2 a_1, -b\rangle = \langle a_2 a_1, b\rangle. \end{aligned}$$

Observing that a_1, a_2 and $a_2 a_1$ form an orthonormal basis of $\mathrm{Im}\,\mathbb{H}$, we have $b \in \mathrm{span}\{a_1, a_2\}$.

By the definition of angle directions,

$$\begin{aligned} &\langle\mathcal{P}_0^\perp x_1, \mathcal{P}_0^\perp x_2\rangle = \langle\mathcal{P}_0^\perp x_1, x_2\rangle = \langle(\mathcal{P}\circ\mathcal{P}_0^\perp)x_1, x_2\rangle \\ =&\langle\mathcal{P}(x_1 - \mathcal{P}_0 x_1), x_2\rangle = \langle x_1, x_2\rangle - \langle(\mathcal{P}\circ\mathcal{P}_0)x_1, x_2\rangle \\ =&\langle x_1, x_2\rangle - \cos^2\theta_1\langle x_1, x_2\rangle = 0, \end{aligned} \tag{3.13}$$

which implies

$$\begin{aligned} 0 &= \langle\sin\theta_1 y_1, \sin\theta_2 y_2\rangle = \sin\theta_1\sin\theta_2\langle a_1\varepsilon, b\varepsilon\rangle \\ &= \sin\theta_1\sin\theta_2\langle a_1, b\rangle \end{aligned}$$

i.e. $\langle a_1, b\rangle = 0$. Therefore $b = a_2$ or $-a_2$.

If $b = a_2$, then (3.11) shows

$$\begin{aligned} x_3 &= (\cos\theta_1\cos\theta_2 a_1 a_2 - \sin\theta_1\sin\theta_2\bar{a}_2 a_1) + (\cos\theta_1\sin\theta_2 a_2 a_1 + \sin\theta_1\cos\theta_2 a_1\bar{a}_2)\varepsilon \\ &= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)a_3 - (\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)a_3\varepsilon \\ &= \cos(\theta_1+\theta_2)a_3 - \sin(\theta_1+\theta_2)a_3\varepsilon. \end{aligned} \tag{3.14}$$

Noting that x_3 is also an angle direction of P relative to $\mathrm{Im}\,\mathbb{H}$, $\theta_3 := \arccos|\cos(\theta_1 + \theta_2)| \in \mathrm{Arg}(P, \mathrm{Im}\,\mathbb{H})$. In other words,

$$\theta_3 = \begin{cases} \theta_1 + \theta_2 & \text{if } \theta_1 + \theta_2 \le \pi/2, \\ \pi - (\theta_1 + \theta_2) & \text{if } \theta_1 + \theta_2 > \pi/2. \end{cases}$$

Otherwise, $b = -a_2$ and (3.11) gives

(3.15)
$$\begin{aligned}
x_3 &= (\cos\theta_1\cos\theta_2 a_1 a_2 - \sin\theta_1\sin\theta_2(-\bar{a}_2)a_1) + (\cos\theta_1\sin\theta_2(-a_2 a_1) + \sin\theta_1\cos\theta_2 a_1\bar{a}_2)\varepsilon\\
&= (\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)a_3 - (-\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)a_3\varepsilon\\
&= \cos(\theta_1-\theta_2)a_3 - \sin(\theta_1-\theta_2)a_3\varepsilon,
\end{aligned}$$

which implies $\theta_3 := \arccos|\cos(\theta_1-\theta_2)| = |\theta_1-\theta_2| \in \mathrm{Arg}(P, \mathrm{Im}\,\mathbb{H})$. Without loss of generality, one can assume $\theta_1 \geq \theta_2$, then $\theta_3 = \theta_1 - \theta_2$. Now we put

$$\begin{aligned}
&\theta'_1 := \theta_2, \quad \theta'_2 := \theta_3, \quad \theta'_3 := \theta_1,\\
&a'_1 := a_2, \quad a'_2 := a_3, \quad a'_3 := a_1,\\
&x'_1 := x_2, \quad x'_2 := x_3, \quad x'_3 := x_1
\end{aligned}$$

and $\varepsilon' := -\varepsilon$, then

$$\begin{aligned}
x'_1 &= \cos\theta'_1 a'_1 + \sin\theta'_1 a'_1\varepsilon',\\
x'_2 &= \cos\theta'_2 a'_2 + \sin\theta'_2 a'_2\varepsilon',\\
x'_3 &= \cos\theta'_3 a'_3 - \sin\theta'_3 a'_3\varepsilon',
\end{aligned} \tag{3.16}$$

which satisfy $\theta'_3 = \theta'_1 + \theta'_2$, $a'_3 = a'_1 a'_2$ and $x'_3 = x'_1 x'_2$.

Altogether, we have shown

Proposition 3.1. *Let P be an associative subspace of Im $\mathbb{O}$, and $0 \leq \theta_1 \leq \theta_2 \leq \theta_3 \leq \pi/2$ be the Jordan angles between P and Im $\mathbb{H}$, then*

$$\theta_3 = \begin{cases} \theta_1+\theta_2 & \text{if } \theta_1+\theta_2 \leq \pi/2,\\ \pi-(\theta_1+\theta_2) & \text{if } \theta_1+\theta_2 > \pi/2. \end{cases} \tag{3.17}$$

Moreover, there exist an orthonormal basis $\{a_1, a_2, a_3\}$ of Im $\mathbb{H}$ satisfying $a_3 = a_1 a_2$, and a unit element $\varepsilon \in \mathbb{H}e$, such that

$$\begin{aligned}
x_1 &:= \cos\theta_1 a_1 + \sin\theta_1 a_1\varepsilon,\\
x_2 &:= \cos\theta_2 a_2 + \sin\theta_2 a_2\varepsilon,\\
x_3 &:= \cos(\theta_1+\theta_2)a_3 - \sin(\theta_1+\theta_2)a_3\varepsilon
\end{aligned} \tag{3.18}$$

are unit angle directions of P relative to Im $\mathbb{H}$, and $x_3 = x_1 x_2$.

3.3. **On the second fundamental form of coassociative submanifolds.** Let M be a 4-dimensional submanifold in Im $\mathbb{O}$. If the normal space at every point of M is associative, then we call M a *coassociative submanifold* (see [18]). Let p be an arbitrary point of M, denote by $0 \leq \theta_1 \leq \theta_2 \leq \theta_3 \leq \pi/2$ the Jordan angles between N_pM (an associative subspace) and Im $\mathbb{H}$, then by Proposition 3.1,

$$\theta_3 = \begin{cases} \theta_1+\theta_2 & \text{if } \theta_1+\theta_2 \leq \pi/2,\\ \pi-(\theta_1+\theta_2) & \text{if } \theta_1+\theta_2 > \pi/2. \end{cases} \tag{3.19}$$

Denote $\{a_1, a_2, a_3\}$ to be the orthonormal basis of Im $\mathbb{H}$ satisfying $a_3 = a_1 a_2$ and ε to be the unit element in $\mathbb{H}e$, such that

$$
\begin{aligned}
\nu_1 &:= \cos\theta_1 a_1 + \sin\theta_1 a_1\varepsilon,\\
\nu_2 &:= \cos\theta_2 a_2 + \sin\theta_2 a_2\varepsilon,\\
\nu_3 &:= \cos(\theta_1+\theta_2) a_3 - \sin(\theta_1+\theta_2) a_3\varepsilon
\end{aligned}
\tag{3.20}
$$

are all unit angle directions of N_pM relative to Im $\mathbb{H}$, and $\nu_3 = \nu_1\nu_2$. Denote

$$
\begin{aligned}
e_1 &:= -\nu_1\varepsilon = \sin\theta_1 a_1 - \cos\theta_1 a_1\varepsilon,\\
e_2 &:= -\nu_2\varepsilon = \sin\theta_2 a_2 - \cos\theta_2 a_2\varepsilon,\\
e_3 &:= -\nu_3\varepsilon = -\sin(\theta_1+\theta_2) a_3 - \cos(\theta_1+\theta_2) a_3\varepsilon,\\
e_4 &:= \varepsilon,
\end{aligned}
\tag{3.21}
$$

then it is easy to check that $\langle e_i, \nu_\alpha\rangle = 0$ and $\langle e_i, e_j\rangle = \delta_{ij}$ for each $1 \le i, j \le 4$ and $1 \le \alpha \le 3$. Hence $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of T_pM. Whenever $\theta_\alpha \in (0, \pi/2)$, let Φ_{p,θ_α} denote the isometric automorphism of $R_{p,\theta_\alpha}M := N_{p,\theta_\alpha}M \oplus T_{p,\theta_\alpha}M$ as in §1.1, then it follows from (1.11) that

$$\sec\theta_1 \mathcal{P}_0^\perp e_1 = \cos\theta_1 e_1 - \sin\theta_1 \Phi_{p,\theta_1}(e_1).$$

Hence

$$
\begin{aligned}
\Phi_{p,\theta_1}(e_1) &= \cot\theta_1 e_1 - \sec\theta_1\csc\theta_1 \mathcal{P}_0^\perp e_1\\
&= \cot\theta_1(\sin\theta_1 a_1 - \cos\theta_1 a_1\varepsilon) - \sec\theta_1\csc\theta_1(-\cos\theta_1 a_1\varepsilon)\\
&= \cos\theta_1 a_1 + \sin\theta_1 a_1\varepsilon\\
&= \nu_1
\end{aligned}
$$

and similarly $\Phi_{p,\theta_2}(e_2) = \nu_2$; in conjunction with (3.19),

$$
\begin{aligned}
\Phi_{p,\theta_3}(e_3) &= \cot\theta_3 e_3 - \sec\theta_3\csc\theta_3 \mathcal{P}_0^\perp e_3\\
&= -\cos\theta_3 a_3 + \mathrm{sgn}\big(\cos(\theta_1+\theta_2)\big)\sin\theta_3 a_3\varepsilon\\
&= \begin{cases} -\nu_3 & \text{if } \theta_1+\theta_2 < \pi/2,\\ \nu_3 & \text{if } \theta_1+\theta_2 > \pi/2. \end{cases}
\end{aligned}
$$

In summary we get a proposition as follows.

Proposition 3.2. *Let M be a coassociative submanifold in Im $\mathbb{O}$, $p \in M$ and $0 \le \theta_1 \le \theta_2 \le \theta_3 \le \pi/2$ be the Jordan angles between N_pM and Im $\mathbb{H}$, then there exist an orthonormal basis $\{\nu_1, \nu_2, \nu_3\}$ of N_pM and an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of T_pM, such that*

(i) For each $1 \le \alpha \le 3$, ν_α (e_α) is an angle direction of N_pM (T_pM) relative to Im $\mathbb{H}$ (or $\mathbb{H}e$), corresponding to the Jordan angle θ_α;

(ii) $e_4 \in \mathbb{H}e$;

(iii) $e_\alpha = -\nu_\alpha e_4$ for each $1 \le \alpha \le 3$;

(iv) $\Phi_{p,\theta_\alpha}(e_\alpha) = \nu_\alpha$ for any $1 \le \alpha \le 2$ satisfying $\theta_\alpha \in (0, \pi/2)$;

(v) $\Phi_{p,\theta_3}(e_3) = \begin{cases} -\nu_3 & \text{if } \theta_1+\theta_2<\pi/2, \\ \nu_3 & \text{if } \theta_1+\theta_2>\pi/2, \\ 0 & \text{if } \theta_1+\theta_2=\pi/2. \end{cases}$

Remark. Here we additionally define $\Phi_{p,0}=\Phi_{p,\pi/2}=0$, as in §2.2.

Now we extend $\{\nu_1,\nu_2,\nu_3\}$ as an orthonormal normal frame field on U, a neighborhood of p, such that $\nabla_v\nu_\alpha=0$ for every $v\in T_pM$. Lemma 3.1 implies $\nu_3(q)=\nu_1(q)\nu_2(q)$ or $-\nu_1(q)\nu_2(q)$ for an arbitrary $q\in U$. Due to the continuity, $\nu_3=\nu_1\nu_2$ on U and differentiating both sides with respect to $e_i\in T_pM$ gives

$$\begin{aligned}-h_{3,ij}e_j &= \overline{\nabla}_{e_i}\nu_3=\overline{\nabla}_{e_i}(\nu_1\nu_2)\\ &=(\overline{\nabla}_{e_i}\nu_1)\nu_2+\nu_1(\overline{\nabla}_{e_i}\nu_2)\\ &=-h_{1,ij}e_j\nu_2-h_{2,ij}\nu_1e_j,\end{aligned}$$

i.e.

$$h_{3,ij}e_j=h_{1,ij}e_j\nu_2+h_{2,ij}\nu_1e_j. \tag{3.22}$$

With the aid of Lemma 3.1, Lemma 3.2 and Proposition 3.2, a straightforward calculation shows

$$\begin{aligned}\text{LHS of (3.22)} &= h_{3,i\alpha}e_\alpha+h_{3,i4}e_4\\ &=-h_{3,i\alpha}\nu_\alpha e_4+h_{3,i4}e_4\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}\text{RHS of (3.22)} =&h_{1,i\alpha}e_\alpha\nu_2+h_{1,i4}e_4\nu_2+h_{2,i\alpha}\nu_1e_\alpha+h_{2,i4}\nu_1e_4\\ =&-h_{1,i\alpha}(\nu_\alpha e_4)\nu_2-h_{1,i4}\nu_2e_4-h_{2,i\alpha}\nu_1(\nu_\alpha e_4)+h_{2,i4}\nu_1e_4\\ =&h_{1,i\alpha}(\nu_\alpha\nu_2)e_4-h_{1,i4}\nu_2e_4-h_{2,i\alpha}(\nu_\alpha\nu_1)e_4+h_{2,i4}\nu_1e_4\\ =&h_{1,i1}\nu_3e_4-h_{1,i2}e_4-h_{1,i3}\nu_1e_4-h_{1,i4}\nu_2e_4\\ &+h_{2,i1}e_4+h_{2,i2}\nu_3e_4-h_{2,i3}\nu_2e_4+h_{2,i4}\nu_1e_4\\ =&(-h_{1,i2}+h_{2,i1})e_4+(-h_{1,i3}+h_{2,i4})\nu_1e_4\\ &+(-h_{1,i4}-h_{2,i3})\nu_2e_4+(h_{1,i1}+h_{2,i2})\nu_3e_4.\end{aligned} \tag{3.24}$$

Comparing with (3.23) and (3.24), we arrive at the following conclusion.

Proposition 3.3. *Let M be a coassociative submanifold in $Im\,\mathbb{O}$, $p\in M$. Let $\{e_i:1\le i\le 4\}$ and $\{\nu_\alpha:1\le\alpha\le 3\}$ be as in Proposition 3.2. Then for each $1\le i\le 4$,*

$$h_{3,i1}=h_{1,i3}-h_{2,i4}, \tag{3.25}$$

$$h_{3,i2}=h_{1,i4}+h_{2,i3}, \tag{3.26}$$

$$h_{3,i3}=-h_{1,i1}-h_{2,i2}, \tag{3.27}$$

$$h_{3,i4}=-h_{1,i2}+h_{2,i1}. \tag{3.28}$$

Here $\{h_{ij}^\alpha:=\langle B_{e_ie_j},\nu_\alpha\rangle(p):1\le i,j\le 4, 1\le\alpha\le 3\}$ are the coefficients of the second fundamental form at p.

3.4. **The characterization of the Lawson-Osserman's cone.** Now we additionaly assume M has CJA relative to Im $\mathbb{H}$. Let $p_0 \in M$, $0 \le \theta_1 \le \theta_2 \le \theta_3 \le \pi/2$ be the Jordan angles between $N_{p_0}M$ and Im $\mathbb{H}$, $\{\nu_1, \nu_2, \nu_3\}$ be the orthonormal basis of $N_{p_0}M$ and $\{e_1, e_2, e_3, e_4\}$ be the orthonormal basis of $T_{p_0}M$, satisfying the properties in Proposition 3.2. Then (2.6) and Lemma 2.3 implies

$$h_{\alpha,\alpha i} = 0 \qquad \forall 1 \le \alpha \le 3, 1 \le i \le 4 \tag{3.29}$$

and substituting it into (3.25)-(3.28) gives

$$h_{1,23} = h_{2,31} = h_{3,12}; \tag{3.30}$$

$$h_{1,22} = -h_{3,24}, \quad h_{1,33} = h_{2,34}, \quad h_{1,44} = -h_{2,34} + h_{3,24}; \tag{3.31}$$

$$h_{2,33} = -h_{1,34}, \quad h_{2,11} = h_{3,14}, \quad h_{2,44} = -h_{3,14} + h_{1,34}; \tag{3.32}$$

$$h_{3,11} = -h_{2,14}, \quad h_{3,22} = h_{1,24}, \quad h_{3,44} = -h_{1,24} + h_{2,14}. \tag{3.33}$$

Furthermore, applying Lemma 2.6 and 2.7 yields the following propositions.

Proposition 3.4. *Let M be a coassociative submanifold in Im $\mathbb{O}$, with CJA relative to Im $\mathbb{H}$. If $g^N \le 2$, $\pi/2 \notin Arg^N$ and $Arg^N \ne \{\arccos(\sqrt{6}/6), \arccos(2/3)\}$, then M has to be affine linear.*

Proof. Let p_0 be an arbitrary point in M, and the notations θ_α, ν_α, e_i, $h_{\alpha,ij}$ are same as above.

Case I. $\theta_1 = 0$ and $\theta_2 = \theta_3 < \pi/2$. Then $g^T = g^N \le 2$ and the equality holds if and only if $\theta_2 \ne 0$. It is well-known that coassociative submanfolds are absolutely area minimizing (see [18] §IV.2.B). By Theorem 2.1, M has to be an open set of an affine 4-plane.

Case II. $\theta_1 = \theta_2 \in (0, \pi/4) \cup (\pi/4, \pi/3)$ and $\theta_3 = \begin{cases} 2\theta_1 & \text{if } \theta_1 < \pi/4, \\ \pi - 2\theta_1 & \text{if } \theta_1 > \pi/4. \end{cases}$ Denote $\theta := \theta_1$, then $\text{Arg}^N = \{\theta, \theta_3\}$, $\text{Arg}^T = \{0, \theta, \theta_3\}$; $T_{p_0,\theta}M = \text{span}\{e_1, e_2\}$ and $N_{p_0,\theta}M = \text{span}\{\nu_1, \nu_2\}$ with $\nu_\alpha = \Phi_\theta(e_\alpha)$ for each $1 \le \alpha \le 2$; $T_{p_0,\theta_3}M = \text{span}\{e_3\}$, $N_{p_0,\theta_3}M = \text{span}\{\nu_3\}$ and

$$\Phi_{\theta_3}(e_3) = \begin{cases} \nu_3 & \text{if } \theta_1 > \pi/4, \\ -\nu_3 & \text{if } \theta_1 < \pi/4; \end{cases}$$

$T_{p_0,0} = \text{span}\{e_4\}$. Lemma 2.2 implies

$$h_{1,22} = h_{2,11} = 0. \tag{3.34}$$

Substituting the above equation into (3.31) and (3.32), we get

$$h_{3,24} = h_{3,14} = 0. \tag{3.35}$$

Applying Lemma 2.1 gives

$$0 = h_{1,23} + h_{2,13} = h_{1,24} + h_{2,14}. \tag{3.36}$$

In conjunction with (3.30), we have

$$h_{1,23} = h_{2,31} = h_{3,12} = 0. \tag{3.37}$$

Let $R_{\theta\sigma}$ and $U_{\theta\sigma}$ be tensors of type $(4,0)$, defined in (2.19) and (2.21), respectively. Then

$$\begin{aligned} R_{\theta\theta_3}(e_1,e_2,e_1,e_2) &= \langle B^{\theta_3}_{e_1e_1}, B^{\theta_3}_{e_2e_2}\rangle - \langle B^{\theta_3}_{e_1e_2}, B^{\theta_3}_{e_2e_1}\rangle \\ &= h_{3,11}h_{3,22} - h_{3,12}h_{3,21} \\ &= -h_{2,14}h_{1,24} = h^2_{1,24}, \end{aligned} \tag{3.38}$$

(3.39)

$$\begin{aligned} U_{\theta\theta_3}(e_1,e_2,e_1,e_2) &= \langle (A^{\Phi_\theta(e_1)}e_1)_{\theta_3}, (A^{\Phi_\theta(e_2)}e_2)_{\theta_3}\rangle - \langle (A^{\Phi_\theta(e_2)}e_1)_{\theta_3}, (A^{\Phi_\theta(e_1)}e_2)_{\theta_3}\rangle \\ &= \langle A^{\nu_1}e_1, e_3\rangle\langle A^{\nu_2}e_2, e_3\rangle - \langle A^{\nu_2}e_1, e_3\rangle\langle A^{\nu_1}e_2, e_3\rangle \\ &= h_{1,13}h_{2,23} - h_{2,13}h_{1,23} = 0 \end{aligned}$$

and

$$\begin{aligned} U_{\theta 0}(e_1,e_2,e_1,e_2) &= \langle (A^{\Phi_\theta(e_1)}e_1)_0, (A^{\Phi_\theta(e_2)}e_2)_0\rangle - \langle (A^{\Phi_\theta(e_2)}e_1)_0, (A^{\Phi_\theta(e_1)}e_2)_0\rangle \\ &= \langle A^{\nu_1}e_1, e_4\rangle\langle A^{\nu_2}e_2, e_4\rangle - \langle A^{\nu_2}e_1, e_4\rangle\langle A^{\nu_1}e_2, e_4\rangle \\ &= h_{1,14}h_{2,24} - h_{2,14}h_{1,24} = h^2_{1,24}. \end{aligned} \tag{3.40}$$

By (2.22),

$$\begin{aligned} 0 &= \sum_{\sigma\in \mathrm{Arg}^N, \sigma\neq\theta} \kappa_{\theta\sigma} R_{\theta\sigma}(e_1,e_2,e_1,e_2) - 3\sum_{\sigma\in \mathrm{Arg}^T, \sigma\neq\theta} \kappa_{\theta\sigma} U_{\theta\sigma}(e_1,e_2,e_1,e_2) \\ &= \kappa_{\theta\theta_3}R_{\theta\theta_3}(e_1,e_2,e_1,e_2) - 3\kappa_{\theta\theta_3}U_{\theta\theta_3}(e_1,e_2,e_1,e_2) - 3\kappa_{\theta 0}U_{\theta 0}(e_1,e_2,e_1,e_2) \\ &= (\kappa_{\theta\theta_3} - 3\kappa_{\theta 0})h^2_{1,24} \end{aligned}$$

where

$$\begin{aligned} \kappa_{\theta\theta_3} - 3\kappa_{\theta 0} &= \frac{\sin 2\theta}{\cos 2\theta - \cos 2\theta_3} - \frac{3\sin 2\theta}{\cos 2\theta - 1} \\ =&\frac{\sin 2\theta}{\cos 2\theta - \cos 4\theta} + \frac{3\sin 2\theta}{1-\cos 2\theta} = \frac{2\cos\theta\sin\theta}{2\sin 3\theta\sin\theta} + \frac{6\cos\theta\sin\theta}{2\sin^2\theta} \\ =&\frac{\cos\theta(\sin\theta + 3\sin 3\theta)}{\sin 3\theta\sin\theta} > 0. \end{aligned}$$

Hence $h_{1,24} = 0$ and moreover

$$h_{3,22} = h_{1,24} = 0, \quad h_{3,11} = -h_{2,14} = h_{1,24} = 0, \quad h_{3,44} = -h_{1,24} + h_{2,14} = 0. \tag{3.41}$$

In conjunction with (3.29), (3.35) and (3.37), we obtain

$$A^{\nu_3} = 0. \tag{3.42}$$

Putting $v = w = e_3$ in (2.23) gives

$$\begin{aligned} \langle (\nabla_{e_3}B)_{e_3e_3}, \Phi_{\theta_3}(e_3)\rangle =&(1/3)\kappa_{\theta_3\theta}\left(\langle B^\theta_{e_3e_3}, B^\theta_{e_3e_3}\rangle + 2|B^\theta_{e_3e_3}|^2\right) \\ &- 2\kappa_{\theta\theta_3}\langle B^\theta_{e_3e_3}, \Phi_\theta(A^{\Phi_{\theta_3}(e_3)}e_3)\rangle \\ =&\kappa_{\theta_3\theta}|B^\theta_{e_3e_3}|^2 = \kappa_{\theta_3\theta}(h^2_{1,33} + h^2_{2,33}) \\ =&\kappa_{\theta_3\theta}(h^2_{2,34} + h^2_{1,34}) \end{aligned} \tag{3.43}$$

(where we have used (3.42), (3.31) and (3.32)). By Lemma 2.7,

$$\begin{aligned}\langle(\nabla_{e_3}B)_{e_4e_4},\Phi_{\theta_3}(e_3)\rangle =& -2\kappa_{\theta\theta_3}\langle B^\theta_{e_3e_4},\Phi_\theta(A^{\Phi_{\theta_3}(e_3)}e_4)_\theta\rangle\\ &+\kappa_{\theta_3\theta}|B^\theta_{e_3e_4}|^2+\kappa_{\theta_3\theta}|(A^{\Phi_{\theta_3}(e_3)}e_4)_\theta|^2+\kappa_{\theta_30}|(A^{\Phi_{\theta_3}(e_3)}e_4)_0|^2\\ =&\kappa_{\theta_3\theta}(h^2_{1,34}+h^2_{2,34}),\end{aligned}\tag{3.44}$$

$$\begin{aligned}\langle(\nabla_{e_3}B)_{e_1e_1},\Phi_{\theta_3}(e_3)\rangle =& -2\kappa_{\theta\theta_3}\langle B^\theta_{e_3e_1},\Phi_\theta(A^{\Phi_{\theta_3}(e_3)}e_1)_\theta\rangle\\ &+\kappa_{\theta_3\theta}|B^\theta_{e_3e_1}|^2+\kappa_{\theta_3\theta}|(A^{\Phi_{\theta_3}(e_3)}e_1)_\theta|^2+\kappa_{\theta_30}|(A^{\Phi_{\theta_3}(e_3)}e_1)_0|^2\\ =&\kappa_{\theta_3\theta}(h^2_{1,31}+h^2_{2,31})=0\end{aligned}\tag{3.45}$$

and similarly

$$\langle(\nabla_{e_3}B)_{e_2e_2},\Phi_{\theta_3}(e_3)\rangle=0.\tag{3.46}$$

Combining (3.43)-(3.46) gives

$$\begin{aligned}0=\langle\nabla_{e_3}H,\Phi_{\theta_3}(e_3)\rangle&=\sum_{i=1}^4\langle(\nabla_{e_3}B)_{e_ie_i},\Phi_{\theta_3}(e_3)\rangle\\ &=2\kappa_{\theta_3\theta}(h^2_{1,34}+h^2_{2,34}),\end{aligned}$$

which forces $h_{1,34}=h_{2,34}=0$ (since $\kappa_{\theta_3\theta}=\frac{\sin 2\theta_3}{\cos 2\theta_3-\cos 2\theta}\neq 0$) and moreover

$$\begin{aligned}h_{1,33}=h_{2,34}=0,\quad h_{1,44}=-h_{2,34}+h_{3,24}=0;\\ h_{2,33}=-h_{1,34}=0,\quad h_{2,44}=-h_{3,14}+h_{1,34}=0.\end{aligned}\tag{3.47}$$

In conjunction with (3.29), (3.34), (3.37), (3.41) and (3.42), we have $B(p_0)=0$. The arbitrariness of p_0 implies $B\equiv 0$, i.e. M is totally geodesic.

Case III. $\theta_1=\theta_2=\theta_3=\pi/3$. Then $g^N=1$, $g^T=2$ and Theorem 2.1 implies M is affine linear.

Case IV. $\theta_2=\theta_3\in(\pi/3,\arccos(\sqrt{6}/6))\cup(\arccos(\sqrt{6}/6),\pi/2)$ and $\theta_1=\pi-2\theta_2$. Denote $\theta:=\theta_2$, then $\mathrm{Arg}^N=\{\theta,\theta_1\}$, $\mathrm{Arg}^T=\{0,\theta,\theta_1\}$; $T_{p_0,\theta}M=\mathrm{span}\{e_2,e_3\}$ and $N_{p_0,\theta}M=\mathrm{span}\{\nu_2,\nu_3\}$ with $\nu_\alpha=\Phi_\theta(e_\alpha)$ for each $2\le\alpha\le 3$; $T_{p_0,\theta_1}M=\mathrm{span}\{e_1\}$ and $N_{p_0,\theta_1}M=\mathrm{span}\{\nu_1\}$ with $\nu_1=\Phi_{\theta_1}(e_1)$; $T_{p_0,0}M=\mathrm{span}\{e_4\}$. Applying Lemma 2.1 and 2.2 gives

$$h_{2,33}=h_{3,22}=0,\quad 0=h_{2,31}+h_{3,21}=h_{2,34}+h_{3,24}.\tag{3.48}$$

Substituting the above equations into (3.30)-(3.33) yields

$$h_{1,23}=h_{2,31}=h_{3,12}=0;\tag{3.49}$$

$$h_{1,22}=-h_{3,24}=h_{2,34}=h_{1,33},\quad h_{1,44}=-2h_{2,34};\tag{3.50}$$

$$h_{1,34}=0,\quad h_{2,11}=h_{3,14}=-h_{2,44};\tag{3.51}$$

$$h_{1,24}=0,\quad h_{3,11}=-h_{2,14}=-h_{3,44}.\tag{3.52}$$

A straightforward calculation shows

$$\begin{aligned}R_{\theta\theta_1}(e_2,e_3,e_2,e_3)&=\langle B^{\theta_1}_{e_2e_2},B^{\theta_1}_{e_3e_3}\rangle-\langle B^{\theta_1}_{e_2e_3},B^{\theta_1}_{e_3e_2}\rangle\\ &=h_{1,22}h_{1,33}-h_{1,23}h_{1,32}=h^2_{2,34},\end{aligned}\tag{3.53}$$

$$
\begin{aligned}
U_{\theta\theta_1}(e_2,e_3,e_2,e_3) &= \langle (A^{\Phi_\theta(e_2)}e_2)_{\theta_1}, A^{\Phi_\theta(e_3)}e_3)_{\theta_1}\rangle - \langle A^{\Phi_\theta(e_3)}e_2)_{\theta_1}, A^{\Phi_\theta(e_2)}e_3)_{\theta_1}\rangle \\
&= h_{2,21}h_{3,31} - h_{3,21}h_{2,31} = 0,
\end{aligned}
\tag{3.54}
$$

$$
\begin{aligned}
U_{\theta 0}(e_2,e_3,e_2,e_3) &= \langle A^{\Phi_\theta(e_2)}e_2)_0, A^{\Phi_\theta(e_3)}e_3)_0\rangle - \langle A^{\Phi_\theta(e_3)}e_2)_0, A^{\Phi_\theta(e_2)}e_3)_0\rangle \\
&= h_{2,24}h_{3,34} - h_{3,24}h_{2,34} = h_{2,34}^2,
\end{aligned}
\tag{3.55}
$$

and then Lemma 2.6 implies

$$
\begin{aligned}
0 &= \sum_{\sigma\in \mathrm{Arg}^N,\sigma\neq 0} \kappa_{\theta\sigma}R_{\theta\sigma}(e_2,e_3,e_2,e_3) - 3\sum_{\sigma\in\mathrm{Arg}^T,\sigma\neq\theta}\kappa_{\theta\sigma}U_{\theta\sigma}(e_2,e_3,e_2,e_3) \\
&= \kappa_{\theta\theta_1}R_{\theta\theta_1}(e_2,e_3,e_2,e_3) - 3\kappa_{\theta\theta_1}U_{\theta\theta_1}(e_2,e_3,e_2,e_3) - 3\kappa_{\theta 0}U_{\theta 0}(e_2,e_3,e_2,e_3) \\
&= (\kappa_{\theta\theta_1} - 3\kappa_{\theta 0})h_{2,34}^2,
\end{aligned}
\tag{3.56}
$$

where

$$
\begin{aligned}
\kappa_{\theta\theta_1} - 3\kappa_{\theta 0} &= \frac{\sin 2\theta}{\cos 2\theta - \cos 2\theta_1} - \frac{3\sin 2\theta}{\cos 2\theta - 1} \\
&= \frac{\sin 2\theta}{\cos 2\theta - \cos 4\theta} + \frac{3\sin 2\theta}{1-\cos 2\theta} = \frac{\cos\theta(\sin\theta + 3\sin 3\theta)}{\sin 3\theta\sin\theta} \\
&= \frac{2\cos\theta(5 - 6\sin^2\theta)}{\sin 3\theta}.
\end{aligned}
\tag{3.57}
$$

Since $\theta \neq \arccos(\sqrt{6}/6)$, $\sin^2\theta \neq 5/6$ and then $\kappa_{\theta\theta_1} - 3\kappa_{\theta 0} \neq 0$. Hence $h_{2,34} = 0$ and moreover

$$
h_{1,22} = h_{1,33} = h_{1,44} = h_{3,24} = h_{2,34} = 0. \tag{3.58}
$$

In conjunction with (3.29), (3.49), (3.51) and (3.52), we have $A^{\nu_1} = 0$. With the aid of Lemma 2.7, one can then proceed as in Case II to deduce that $B(p_0) = 0$. Since p_0 is arbitrary, M has to be affine linear.

□

Proposition 3.5. *Let M be a coassociative submanifold of Im $\mathbb{O}$. Assume M has CJA relative to Im $\mathbb{H}$, and Arg$^N = \{\arccos(\sqrt{6}/6), \arccos(2/3)\}$, then either M is affine linear, or there exists $a_0 \in Sp_1$, such that M is a translate of an open subset of $M(a_0)$. Here $M(a_0)$ denotes the Lawson-Osserman's cone, see (1.30).*

Proof. Let $\theta_1 := \arccos(2/3)$, $\theta_2 = \theta_3 := \arccos(\sqrt{6}/6)$ and $\theta := \theta_2$. For an arbitrary point $p_0 \in M$, let $\{e_1,e_2,e_3,e_4\}$ be an orthonormal tangent frame field and $\{\nu_1,\nu_2,\nu_3\}$ be an orthonormal normal frame field on U, a neighborhood of p_0, such that for any $p \in M$, $e_i(p)$ and $\nu_\alpha(p)$ satisfy the properties in Proposition 3.2. In particular, $\nu_\alpha = \Phi_\alpha(e_\alpha)$ for each $1 \le \alpha \le 3$. With the aid of Lemma 2.1, Lemma 2.2 and Proposition 3.3, one can proceed as above to get some pointwise relations between the coefficients of the second fundamental form, see (3.29), (3.48)-(3.52). Denote

$$
h := h_{1,22}, \tag{3.59}
$$

then h can be seen as a smooth function on U, and

$$
h_{1,33} = h_{2,34} = h, \quad h_{3,24} = -h, \quad h_{1,44} = -2h. \tag{3.60}
$$

Step I. Prove

$$h_{2,11} = h_{3,11} = h_{2,14} = h_{3,14} = h_{2,44} = h_{3,44} = 0. \tag{3.61}$$

By (2.23),

$$\begin{aligned}\langle (\nabla_{e_1} B)_{e_1 e_1}, \Phi_{\theta_1}(e_1)\rangle =&(1/3)\kappa_{\theta_1\theta}\left(\langle B^\theta_{e_1e_1}, B^\theta_{e_1e_1}\rangle + 2|B^\theta_{e_1e_1}|^2\right)\\ &- 2\kappa_{\theta\theta_1}\langle B^\theta_{e_1e_1}, \Phi_\theta(A^{\Phi_{\theta_1}(e_1)}e_1)\rangle\\ =&\kappa_{\theta_1\theta}(h^2_{2,11} + h^2_{3,11}).\end{aligned}\tag{3.62}$$

Applying Lemma 2.7, we have

$$\begin{aligned}\langle (\nabla_{e_1} B)_{e_4 e_4}, \Phi_{\theta_1}(e_1)\rangle =& - 2\kappa_{\theta\theta_1}\langle B^\theta_{e_1e_4}, \Phi_\theta(A^{\Phi_{\theta_1}}e_4)_\theta\rangle\\ &+ \kappa_{\theta_1\theta}|B^\theta_{e_1e_4}|^2 + \kappa_{\theta_1\theta}|(A^{\Phi_{\theta_1}(e_1)}e_4)_\theta|^2 + \kappa_{\theta_1 0}|(A^{\Phi_{\theta_1}(e_1)}e_4)_0|^2\\ =& - 2\kappa_{\theta\theta_1}(h_{2,14}h_{1,42} + h_{3,14}h_{1,43})\\ &+ \kappa_{\theta_1\theta}(h^2_{2,14} + h^2_{3,14}) + \kappa_{\theta_1\theta}(h^2_{1,42} + h^2_{1,43}) + \kappa_{\theta_1 0}h^2_{1,44}\\ =&\kappa_{\theta_1\theta}(h^2_{2,14} + h^2_{3,14}) + 4\kappa_{\theta_1 0}h^2,\end{aligned}\tag{3.63}$$

$$\begin{aligned}\langle (\nabla_{e_1} B)_{e_2 e_2}, \Phi_{\theta_1}(e_1)\rangle =& - 2\kappa_{\theta\theta_1}\langle B^\theta_{e_1e_2}, \Phi_\theta(A^{\Phi_{\theta_1}}e_2)_\theta\rangle\\ &+ \kappa_{\theta_1\theta}|B^\theta_{e_1e_2}|^2 + \kappa_{\theta_1\theta}|(A^{\Phi_{\theta_1}(e_1)}e_2)_\theta|^2 + \kappa_{\theta_1 0}|(A^{\Phi_{\theta_1}(e_1)}e_2)_0|^2\\ =& - 2\kappa_{\theta\theta_1}(h_{2,12}h_{1,22} + h_{3,12}h_{1,23})\\ &+ \kappa_{\theta_1\theta}(h^2_{2,12} + h^2_{3,12}) + \kappa_{\theta_1\theta}(h^2_{1,22} + h^2_{1,23}) + \kappa_{\theta_1 0}h^2_{1,24}\\ =&\kappa_{\theta_1\theta}h^2_{1,22} = \kappa_{\theta_1\theta}h^2\end{aligned}\tag{3.64}$$

and similarly

$$\langle (\nabla_{e_1} B)_{e_3 e_3}, \Phi_{\theta_1}(e_1)\rangle = \kappa_{\theta_1\theta}h^2_{1,33} = \kappa_{\theta_1\theta}h^2. \tag{3.65}$$

Adding (3.62)-(3.65) gives

$$\begin{aligned}0 = \langle \nabla_{e_1} H, \Phi_{\theta_1}(e_1)\rangle &= \sum_{i=1}^4 \langle (\nabla_{e_1} B)_{e_i e_i}, \Phi_{\theta_1}(e_1)\rangle\\ &= \kappa_{\theta_1\theta}(h^2_{2,11} + h^2_{3,11} + h^2_{2,14} + h^2_{3,14}) + 2(2\kappa_{\theta_1 0} + \kappa_{\theta_1\theta})h^2,\end{aligned}$$

where

$$\begin{aligned}2\kappa_{\theta_1 0} + \kappa_{\theta_1\theta} &= \frac{2\sin 2\theta_1}{\cos 2\theta_1 - 1} + \frac{\sin 2\theta_1}{\cos 2\theta_1 - \cos 2\theta}\\ &=\frac{2\sin 2\theta_1}{\cos 2\theta_1 - 1} + \frac{\sin 2\theta_1}{\cos 2\theta_1 + \cos\theta_1} = \frac{\sin 2\theta_1\left[2(\cos 2\theta_1 + \cos\theta_1) + \cos 2\theta_1 - 1\right]}{(\cos 2\theta_1 - 1)(\cos 2\theta_1 + \cos\theta_1)}\\ &=\frac{2\sin 2\theta_1(3\cos\theta_1 - 2)(\cos\theta_1 + 1)}{(\cos 2\theta_1 - 1)(\cos 2\theta_1 + \cos\theta_1)} = 0.\end{aligned}\tag{3.66}$$

Hence $h_{2,11} = h_{3,11} = h_{2,14} = h_{3,14} = 0$ and substituting it into (3.51)-(3.52) implies $h_{2,44} = h_{3,44} = 0$.

Step II. Calculation of the connection coefficients.

Denote

$$\Gamma_{ij}^k := \langle \nabla_{e_i} e_j, e_k \rangle, \quad \bar{\Gamma}_{i\alpha}^\beta := \langle \nabla_{e_i} \nu_\alpha, \nu_\beta \rangle. \tag{3.67}$$

Then differentiating both sides of $\langle e_i, e_k \rangle = \delta_{jk}$ with respect to e_i gives $\Gamma_{ij}^k + \Gamma_{ik}^j = 0$. In particular, $\Gamma_{ij}^j = 0$. Similarly $\bar{\Gamma}_{i\alpha}^\beta + \bar{\Gamma}_{i\beta}^\alpha = 0$ and especially $\bar{\Gamma}_{i\alpha}^\alpha = 0$.

Based on Lemma 2.4, a direct calculation shows

$$\begin{aligned}\Gamma_{i1}^4 &= (S_{\theta_1 0})_{e_1 e_4}(e_i) = \kappa_{0\theta_1} \langle B_{e_i e_1}, \Phi_0(e_4) \rangle - \kappa_{\theta_1 0} \langle B_{e_i e_4}, \Phi_{\theta_1}(e_1) \rangle \\ &= -\kappa_{\theta_1 0} h_{1,i4},\end{aligned} \tag{3.68}$$

$$\begin{aligned}\Gamma_{i1}^2 &= (S_{\theta_1 \theta})_{e_1 e_2}(e_i) = \kappa_{\theta\theta_1} \langle B_{e_i e_1}, \Phi_\theta(e_2) \rangle - \kappa_{\theta_1 \theta} \langle B_{e_i e_2}, \Phi_{\theta_1}(e_1) \rangle \\ &= \kappa_{\theta\theta_1} h_{2,i1} - \kappa_{\theta_1\theta} h_{1,i2},\end{aligned} \tag{3.69}$$

$$\begin{aligned}\Gamma_{i2}^4 &= (S_{\theta 0})_{e_2 e_4}(e_i) = \kappa_{0\theta} \langle B_{e_i e_2}, \Phi_0(e_4) \rangle - \kappa_{\theta 0} \langle B_{e_i e_4}, \Phi_\theta(e_2) \rangle \\ &= -\kappa_{\theta 0} h_{2,i4}\end{aligned} \tag{3.70}$$

and similarly

$$\Gamma_{i1}^3 = (S_{\theta_1\theta})_{e_1 e_3}(e_i) = \kappa_{\theta\theta_1} h_{3,i1} - \kappa_{\theta_1\theta} h_{1,i3}, \tag{3.71}$$

$$\Gamma_{i3}^4 = (S_{\theta 0})_{e_3 e_4}(e_i) = -\kappa_{\theta 0} h_{3,i4}. \tag{3.72}$$

By Lemma 2.5,

$$\begin{aligned}\bar{\Gamma}_{21}^3 &= (S_{\theta_1\theta}^N)_{\nu_1 \nu_3}(e_2) = \kappa_{\theta_1\theta} \langle B_{e_2, \Phi_{\theta_1}(\nu_1)}, \nu_3 \rangle - \kappa_{\theta\theta_1} \langle B_{e_2, \Phi_\theta(\nu_3)}, \nu_1 \rangle \\ &= \kappa_{\theta\theta_1} h_{1,23} - \kappa_{\theta_1\theta} h_{3,21} = 0.\end{aligned} \tag{3.73}$$

Step III. Proof that the angle lines with respect to θ_1, i.e. integral curves of the vector field e_1, must be straight lines in Euclidean space.

This is equivalent to $\overline{\nabla}_{e_1} e_1 = 0$ holding everywhere, which follows from the following straightforward calculation.

$$\begin{aligned}\overline{\nabla}_{e_1} e_1 = B_{e_1 e_1} + \nabla_{e_1} e_1 = h_{\alpha,11} \nu_\alpha + \Gamma_{11}^i e_i &= \sum_{i=2}^3 \Gamma_{11}^i e_i + \Gamma_{11}^4 e_4 \\ &= \sum_{i=2}^3 (\kappa_{\theta\theta_1} h_{i,11} - \kappa_{\theta_1\theta} h_{1,1i}) e_i - \kappa_{\theta_1 0} h_{1,14} e_4 = 0.\end{aligned}$$

Step IV. Proof that there exists a hypersurface N of U, such that $p_0 \in N$ and $e_1(p) \perp T_p N$ for every $p \in N$.

By the Frobenius theorem, it suffices to prove that the subbundle $e_1^\perp$ of TU is integrable; more precisely, given arbitrary smooth sections X, Y of $e_1^\perp$, $[X, Y]$ takes values in $e_1^\perp$ as well.

Now we write $X = \sum_{i=2}^4 X^i e_i$ and $Y = \sum_{j=2}^4 Y^j e_j$, then

$$[X, Y] = X^i Y^j [e_i, e_j] + X^i (\nabla_{e_i} Y^j) e_j - Y^j (\nabla_{e_j} X^i) e_i$$

and hence

$$\langle [X,Y], e_1\rangle = X^i Y^j \langle [e_i, e_j], e_1\rangle.$$

Hence it is necessary and sufficient for us to show $\langle [e_i, e_j], e_1\rangle = 0$ for any $2 \le i < j \le 4$.

Since ∇ is torsion-free,

$$\begin{aligned}\langle [e_2, e_3], e_1\rangle =& \langle \nabla_{e_2} e_3, e_1\rangle - \langle \nabla_{e_3} e_2, e_1\rangle = \Gamma_{23}^1 - \Gamma_{32}^1 = -\Gamma_{21}^3 + \Gamma_{31}^2\\ =& -(\kappa_{\theta\theta_1} h_{3,21} - \kappa_{\theta_1\theta} h_{1,23}) + (\kappa_{\theta\theta_1} h_{2,31} - \kappa_{\theta_1\theta} h_{1,32})\\ =& 0,\end{aligned}$$

$$\begin{aligned}\langle [e_2, e_4], e_1\rangle =& \langle \nabla_{e_2} e_4, e_1\rangle - \langle \nabla_{e_4} e_2, e_1\rangle = \Gamma_{24}^1 - \Gamma_{42}^1 = -\Gamma_{21}^4 + \Gamma_{41}^2\\ =& \kappa_{\theta_1 0} h_{1,24} + (\kappa_{\theta\theta_1} h_{2,41} - \kappa_{\theta_1\theta} h_{1,42})\\ =& 0\end{aligned}$$

and similarly

$$\langle [e_3, e_4], e_1\rangle = \kappa_{\theta_1 0} h_{1,34} + (\kappa_{\theta\theta_1} h_{3,41} - \kappa_{\theta_1\theta} h_{1,43}) = 0.$$

Then the claim is proved.

Without loss of generality, we can assume that the closure of N is contained in U. Then there exists $\delta > 0$, such that $\mathbf{X}(p) + te_1 \in U$ for every $p \in N$ and any $t \in (-\delta, \delta)$, where $\mathbf{X}(p)$ denotes the position vector of p in Im $\mathbb{O}$. Define $\phi : N \times (-\delta, \delta) \to U$

$$(p, t) \mapsto \mathbf{X}(p) + te_1, \tag{3.74}$$

then ϕ is a diffeomorphism between $N \times (-\delta, \delta)$ and a neighborhood of p_0 in M, which is denoted by W.

Step V. The function h defined in (3.59) is constant on N.

Applying the Codazzi equations,

$$\begin{aligned}\nabla_{e_4} h &= \nabla_{e_4} h_{1,22} = \nabla_{e_4} \langle B_{e_2 e_2}, \nu_1\rangle\\ &= \langle (\nabla_{e_4} B)_{e_2 e_2}, \nu_1\rangle + 2\Gamma_{42}^i h_{1,2i} + \Gamma_{41}^\alpha h_{\alpha,22}\\ &= \langle (\nabla_{e_4} B)_{e_2 e_2}, \nu_1\rangle = \langle (\nabla_{e_2} B)_{e_2 e_4}, \nu_1\rangle\\ &= \nabla_{e_2} h_{1,24} - \Gamma_{22}^i h_{1,i4} - \Gamma_{24}^i h_{1,2i} - \bar\Gamma_{21}^\alpha h_{\alpha,24}\\ &= 2\Gamma_{22}^4 h - \Gamma_{24}^2 h + \bar\Gamma_{21}^3 h = 3\Gamma_{22}^4 h\\ &= -3\kappa_{\theta 0} h_{2,24} h = 0,\end{aligned}$$

$$\begin{aligned}\nabla_{e_2} h &= \nabla_{e_2} h_{1,33} = \nabla_{e_2} \langle B_{e_3 e_3}, \nu_1\rangle\\ &= \langle (\nabla_{e_2} B)_{e_3 e_3}, \nu_1\rangle + 2\Gamma_{23}^i h_{1,3i} + \bar\Gamma_{21}^\alpha h_{\alpha,33}\\ &= \langle (\nabla_{e_2} B)_{e_3 e_3}, \nu_1\rangle = \langle (\nabla_{e_3} B)_{e_2 e_3}, \nu_1\rangle\\ &= \nabla_{e_3} h_{1,23} - \Gamma_{32}^i h_{1,i3} - \Gamma_{33}^i h_{1,2i} - \bar\Gamma_{31}^\alpha h_{\alpha,23}\\ &= -\Gamma_{32}^3 h - \Gamma_{33}^2 h = 0\end{aligned}$$

and similarly

$$\nabla_{e_3} h = \nabla_{e_3} h_{1,22} = -\Gamma_{23}^2 h - \Gamma_{22}^3 h = 0.$$

Hence $\nabla h \equiv 0$ on N. Without loss of generality, we can assume $h|_N \equiv h_0$, with h_0 a nonnegative constant.

Step VI. W is a cone whenever $h_0 > 0$.

Define $\psi : N \to \text{Im }\mathbb{O}$

$$\psi(p) = \mathbf{X}(p) + R_0 e_1(p),$$

where R_0 is a constant to be chosen. Then

$$\begin{aligned}
\psi_* e_i &= e_i + R_0 \overline{\nabla}_{e_i} e_1 \\
&= e_i + R_0\big(B_{e_i e_1} + \nabla_{e_i} e_1\big) \\
&= e_i + R_0 \Gamma_{i1}^j e_j \\
&= e_i + R_0\Big[\sum_{j=2}^{3} (\kappa_{\theta\theta_1} h_{j,i1} - \kappa_{\theta_1\theta} h_{1,ij}) e_j - \kappa_{\theta_1 0} h_{1,i4} e_4\Big]
\end{aligned}$$

for each $2 \le i \le 4$. More precisely,

$$\begin{aligned}
\psi_* e_2 &= (1 - R_0 \kappa_{\theta_1\theta} h_{1,22}) e_2 = (1 - R_0 \kappa_{\theta_1\theta} h_0) e_2, \\
\psi_* e_3 &= (1 - R_0 \kappa_{\theta_1\theta} h_{1,33}) e_3 = (1 - R_0 \kappa_{\theta_1\theta} h_0) e_3, \\
\psi_* e_4 &= (1 - R_0 \kappa_{\theta_1 0} h_{1,44}) e_4 = (1 + 2R_0 \kappa_{\theta_1 0} h_0) e_4.
\end{aligned} \tag{3.75}$$

Now we put

$$R_0 := (\kappa_{\theta_1\theta} h_0)^{-1},$$

then combining (3.75) and (3.66) implies $\psi_* e_i = 0$ for each $2 \le i \le 4$. Hence ψ is a constant map on N. Without loss of generality, we can assume $\psi \equiv 0$, i.e. $F(p) = -R_0 e_1(p)$ for every $p \in N$. In other words, N lies in the Euclidean sphere centered at 0 and of radius R_0, and an arbitrary normal line of N, i.e. $\{F(p) + t e_1 : t \in \mathbb{R}\}$ with $p \in N$, must go through the origin. Therefore W is a cone.

Step VII. M is an open subset of $M(a_0)$ provided that $h_0 > 0$ and $\psi \equiv 0$.

Let

$$S := \{x \in \text{Im }\mathbb{O} : |x| = R_0, |\mathcal{P}_0^\perp x| = \cos\theta_1 R_0\} \tag{3.76}$$

be a submanifold of Im $\mathbb{O}$. For any $x \in S$, there exist a unit element $b \in \text{Im }\mathbb{H}$ and a unit element $\varepsilon \in \mathbb{H}e$, such that

$$x = R_0(-\sin\theta_1 b + \cos\theta_1 b\varepsilon).$$

Define

$$E_x = \mathbb{R}\varepsilon \oplus \{\sin\theta c - \cos\theta c\varepsilon : c \in \text{Im }\mathbb{H}, \langle b, c\rangle = 0\}, \tag{3.77}$$

then E_x is a 3-dimensional subspace of $T_x S$. Furthermore

$$E := \{E_x : x \in S\} \tag{3.78}$$

is a 3-dimensional distribution on S.

For any $p \in N$, $e_1(p)$ is a unit tangent angle direction associated to θ_1. Hence there exist $b \in \text{Im } \mathbb{H}$ and $\varepsilon \in \mathbb{H}e$ satisfying $|b| = |\varepsilon| = 1$, such that

$$e_1(p) = \sin\theta_1 b - \cos\theta_1 b\varepsilon.$$

Moreover,

$$\begin{aligned}\mathbf{X}(p) &= \psi(p) - R_0 e_1(p) = -R_0 e_1(p)\\ &= R_0(-\sin\theta_1 b + \cos\theta_1 b\varepsilon).\end{aligned}$$

Therefore $N \subset S$.

Denote

$$\begin{aligned}\nu_1 :=& \Phi_{\theta_1}(e_1) = (-\tan\theta_1 \mathcal{P}_0^{\perp} + \cot\theta_1 \mathcal{P}_0)e_1\\ =& \cos\theta_1 b + \sin\theta_1 b\varepsilon,\end{aligned}$$

then ν_1 is a unit angle direction of N_pM with respect to Im $\mathbb{H}$. On the other hand, Proposition 3.1 implies the existence of an orthonormal basis $\{b_1, b_2, b_3\}$ of Im $\mathbb{H}$ satisfying $b_3 = b_1 b_2$ and a unit element $\varepsilon' \in \mathbb{H}e$, such that

$$\nu'_\alpha := \cos\theta_\alpha b_\alpha + \sin\theta_\alpha b_\alpha \varepsilon' \qquad \forall 1 \le \alpha \le 3$$

are all unit angle directions of N_pM relative to Im $\mathbb{H}$. Since $m_{\theta_1} = 1$, $\nu'_1 = \pm\nu_1$, and then one can assume $b_1 = b$, $\varepsilon' = \varepsilon$ without loss of generality, which implies

$$N_pM = \mathbb{R}\nu_1 \oplus \{\cos\theta c + \sin\theta c\varepsilon : c \in \text{Im } H, \langle b, c\rangle = 0\}.$$

Noting that $T_pN \perp N_pM$ and $T_pN \perp e_1$, it is easy to deduce that $T_pN = E_p$, i.e. N is an integral manifold of E.

For any $a \in \text{Sp}_1$, $M(a)$ is a coassociative cone, which has CJA with $\text{Arg}^T = \{\theta_1, \theta, 0\}$, and each ray is an angle line with respect to θ_1. As above, one can show that $M(a) \cap B(R_0) \subset S$ and that it is also an integral manifold of E.

Now we write

$$\mathbf{X}(p_0) = R_0(\sin\theta_1 b_0 + \cos\theta_1 c_0 e) = (2/3)R_0\big[(\sqrt{5}/2)b_0 + c_0 e\big] \tag{3.79}$$

with $b_0 \in \text{Im } \mathbb{H}, c_0 \in \mathbb{H}$ satisfying $|b_0| = |c_0| = 1$. Then choosing

$$a_0 := c_0 b_0 \bar{c}_0, \quad q_0 := \bar{c}_0 \tag{3.80}$$

gives

$$\mathbf{X}(p_0) = (2/3)R_0\big[(\sqrt{5}/2)q_0 a_0 \bar{q}_0 + \bar{q}_0 e\big] \in M(a_0).$$

Therefore N and $M(a_0) \cap B(R_0)$ are both integral manifolds of E. Since $M(a_0) \cap B(R_0)$ is complete, applying the Frobenius theorem implies $N \subset M(a_0) \cap B(R_0)$, and hence $W \subset M(a_0)$. Finally, because minimal submanfolds in Euclidean space are analytic manifolds, M has to be an open subset of $M(a_0)$.

Step VIII. M is affine linear whenever $h_0 = 0$.

Firstly, $h_0 = 0$ implies $B \equiv 0$ on N. Denote by $\tilde{B}$ the second fundamental form of N in Im $\mathbb{O}$, then

$$\langle \tilde{B}_{e_i e_j}, \nu_\alpha\rangle = \langle \overline{\nabla}_{e_i} e_j, \nu_\alpha\rangle = \langle B_{e_i e_j}, \nu_\alpha\rangle = 0$$

for any $2 \le i, j \le 4$ and $1 \le \alpha \le 3$,

$$\begin{aligned}\langle \tilde{B}_{e_i e_j}, e_1 \rangle = \langle \overline{\nabla}_{e_i} e_j, e_1 \rangle = \langle \nabla_{e_i} e_j, e_1 \rangle = \Gamma^1_{ij} \\ = -\Gamma^j_{i1} = -(\kappa_{\theta\theta_1} h_{j,i1} - \kappa_{\theta_1\theta} h_{1,ij}) = 0\end{aligned}$$

for any $2 \le j \le 3$ and

$$\langle \tilde{B}_{e_i e_4}, e_1 \rangle = \Gamma^1_{i4} = -\Gamma^4_{i1} = \kappa_{\theta_1 0} h_{1,i4} = 0.$$

Thus $\tilde{B} \equiv 0$, i.e. N is totally geodesic.

Since

$$\begin{aligned}\overline{\nabla}_{e_i} e_1 &= \sum_{j=2}^4 \langle \nabla_{e_i} e_1, e_j \rangle e_j + B_{e_i e_1} \\ &= \sum_{j=2}^4 \Gamma^j_{i1} e_j + B_{e_i e_1} = 0\end{aligned}$$

for each $2 \le i \le 4$, e_1 is parallel along N. Therefore W is an open subset of an affine linear subspace of Im $\mathbb{O}$. Due to the analyticity of minimal submanifolds, M has to be affine linear. And the proof is completed.

□

Proposition 3.4 and Proposition 3.5 together imply the following theorem.

Theorem 3.1. *Let M be a coassociative submanifold in Im $\mathbb{O}$. Assume M has CJA relative to Im $\mathbb{H}$. If $g^N \le 2$ and $\pi/2 \notin Arg^N$, then either M is affine linear, or there exists $a_0 \in Sp_1$ and $w_0 \in$ Im $\mathbb{O}$, such that M is an open subset of*

$$M(a_0, w_0) := \{r\big[(\sqrt{5}/2) q a_0 \bar{q} + \bar{q} e\big] + w_0 : q \in Sp_1, r \in \mathbb{R}^+\}.$$

In other words, M is a translate of a portion of the Lawson-Osserman's cone.

As at the end of Section 2, we have a corollary.

Corollary 3.1. *Let D be an open domain of $\mathbb{H}$ and $f : D \to$ Im $\mathbb{H}$. If $M =$ graph f is a coassociative submanifold with CJA relative to Im $\mathbb{H}$, and $g^N \le 2$, then f is either an affine linear function or $f(x) = \eta(x - x_0) + y_0$, where $x_0 \in \mathbb{H}$, $y_0 \in$ Im $\mathbb{H}$ and*

$$\eta(x) = \frac{\sqrt{5}}{2|x|} \bar{x} \varepsilon x$$

with ε an arbitrary unit element in Im $\mathbb{H}$.

This is the Theorem 1.2 in §1.5.

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